The Root Atlas of the Modular Neighborhood

Surds, Floors, and the Level Law of the Modular Surface's Immediate Covering Neighborhood

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1. Introduction

1.1 The neighborhood, and the question

Quadratic surds arise on the modular surface by two routes that share no step. A cone point manufactures one out of its own order, through a sine; a closed geodesic manufactures one out of its trace, through a discriminant. The two mechanisms are unrelated, and they return the same numbers often enough to invite a story. The half-product taken over the order-5 cone point of the Hecke group $G(5)$ is $\sqrt{5}$, and so is the eigenvalue gap of the shortest closed geodesic of the modular surface. The order-3 cone point of the modular surface returns $\sqrt{3}$, and the trace-4 class returns a surd generating the same field $\mathbb{Q}(\sqrt{3})$. The order-4 cone point of $G(4)$ returns $\sqrt{2}$, and the trace-6 class of $\mathrm{PSL}(2,\mathbb{Z})$ returns surds of $\mathbb{Q}(\sqrt{2})$. This paper asks what connects such pairs, and answers in the only way available: by naming the mechanism at every occurrence, by drawing a connection only where a derivation supplies one, and by proving that the region in which the question can be posed at all is finite and closed.

The objects are three families. The first is the modular group $\mathrm{PSL}(2,\mathbb{Z})$ and its surface $X(1) = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ — the conjugacy classes of that group, their traces, its two cone points, and the closed geodesics they carry. The second is the family of Hecke triangle groups $G(q)$, generated by $z \mapsto -1/z$ and $z \mapsto z + \lambda_q$ with

$$\lambda_q \;=\; 2\cos(\pi/q), \qquad q \geq 3, \tag{1.1}$$

so that $G(3) = \mathrm{PSL}(2,\mathbb{Z})$ and $G(4), G(5), G(6)$ are the surfaces immediately next door. The third is reached from $X(1)$ in the arithmetic direction rather than the geometric one: for squarefree p the Hecke congruence subgroup $\Gamma_0(p)$ has finite index in $\mathrm{PSL}(2,\mathbb{Z})$, so $X_0(p)$ covers $X(1)$, and adjoining the Fricke involution passes to a quotient of that cover,

$$w_p \;=\; \frac{1}{\sqrt{p}}\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}, \qquad \Gamma_0(p)^+ \;=\; \big\langle\, \Gamma_0(p),\, w_p \,\big\rangle. \tag{1.2}$$

The group $\Gamma_0(p)^+$ is the disjoint union of the two cosets $\Gamma_0(p)$ and $w_p\,\Gamma_0(p)$, and its traces are graded accordingly:

$$\Theta\big(\Gamma_0(p)\big) \;\subseteq\; \mathbb{Z}, \qquad \Theta\big(w_p\,\Gamma_0(p)\big) \;\subseteq\; \sqrt{p}\cdot\mathbb{Z}, \tag{1.3}$$

where Θ denotes the set of traces, taken up to sign. Both containments in (1.3) quantify over the whole group; no ball and no enumeration bound enters either. That is what makes several of the results below theorems rather than surveys, and it is why the two families are treated together.

Two questions organize what follows. The first is whether occurrences that share a value share a mechanism. The second is whether the arithmetic that a closed geodesic of $X(1)$ carries can be read one level down, at half its length — and if so, at which level. The answers are, in order: mostly no, with the exceptions named and derived rather than observed; and yes, on a family that the discriminant itself selects, at a level the discriminant itself determines.

Table 1.1. *The four stages. The trace field is $\mathbb{Q}(\lambda_q)$ and the invariant trace field $\mathbb{Q}(\lambda_q^2)$; arithmeticity is Takeuchi's classification [1], carried with its routes in §4. The last two rows are proved in §5 (Theorems 5.3 and 5.5) for $G(4)$, $G(5)$, $G(6)$, and for $X(1)$ in §8 (Theorem 8.1 with Remark 8.9).*

	$X(1) = G(3)$	$G(4)$	$G(5)$	$G(6)$
signature	$(2, 3, \infty)$	$(2, 4, \infty)$	$(2, 5, \infty)$	$(2, 6, \infty)$
λ_q	1	$\sqrt{2}$	φ	$\sqrt{3}$
trace field	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{3})$
invariant trace field	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}$
arithmetic	yes	yes	**no**	yes
area	$\pi/3$	$\pi/2$	$3\pi/5$	$2\pi/3$
cone orders	$\{2, 3\}$	$\{2, 4\}$	$\{2, 5\}$	$\{2, 6\}$
minimal hyperbolic trace	3	$2\sqrt{2}$	2φ	$2\sqrt{3}$
systole	$4 \log \varphi =$ 1.92485 …	$2 \operatorname{arccosh} \sqrt{2} =$ 1.76275 …	$2 \operatorname{arccosh} \varphi =$ 2.12255 …	$2 \operatorname{arccosh} \sqrt{3} =$ 2.29243 …

1.2 The domain, and its edges

The domain of this paper is the four surfaces of Table 1.1 together with the Fricke groups $\Gamma_0(p)^+$ at squarefree level, and the arithmetic those objects carry: binary quadratic forms of discriminant t^2-4, the real quadratic fields they generate, fundamental units and the norms of those units, narrow class numbers, and the closed geodesics indexed by all of these. The concentration on $q \in \{4, 5, 6\}$ inside the Hecke family is not a choice of scope but a theorem (§2); one further Hecke group, $G(10)$, is admitted at a single atlas entry and marked as a declared extension there. Analytic apparatus — L-functions, the trace formula — appears only where an atlas entry names it as the mechanism of an occurrence (§2.1, §2.3) and is not developed. Discrete groups outside these families are quantified over at exactly one place, §8.6, where two theorems range over all discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$; the quantifier is stated in the theorems themselves. Elsewhere the ambient groups $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{GL}(2, \mathbb{Z})$ appear only as the setting against which a property of the discrete families is contrasted (§8.5, §10.1).

The domain has edges, and they are catalogued in §11 rather than smoothed over here. Four of them indicate the shape of the rest. The integer traces of $G(5)$ are $\{0, \pm 2\}$ over a census exhaustive to syllable depth 12, and that statement is not proved; §11 carries with it a theorem showing that no separation or interval argument can ever close it, however large the census (so the edge is a named obstruction, not an unfinished computation). The list of 25 levels at which a Fricke floor's own element realizes its systole is verified exhaustively for $p \leq 20\,000$, and its completeness over all p requires an effective bound that is not supplied here (§6.4). Whether the absence of mixed traces $a + b\sqrt{2}$ from $G(4)$ is a structural feature of the even-q stages or an artefact of the enumerations that exhibited it is open (§8.4). And at $G(5)$ the order-level partner of reciprocity fails in both directions, leaving an archimedean implication whose converse is false and whose mechanism is not settled (§7.4).

Two disciplines govern how the results below are stated. First, **no completeness claim in this paper is absolute**: wherever a list, a census or a classification is called complete, the sentence that calls it so also says what it is complete over — a range of traces, a range of levels, a set of orders, a syllable depth, a population of discriminants, or, in the places where the coset grading (1.3) or a normal form makes it available, the whole group. A bound stated inline is a statement about recall, not a hedge, and the two are kept apart throughout. Second, the paper draws no conclusion about *why* the two families of mechanisms should return the same numbers as often as they do; that question is not addressed. Where a sentence below reads as an answer to it, the sentence overstates its section, and the section is what governs.

1.3 The results

§2 — the atlas, and the organizing theorem. The catalogue of surd occurrences is indexed by occurrence and not by value, with the manufacturing mechanism named at every entry and a connection drawn only where a derivation supplies one. It is then closed on the right by Theorem 2.10: the trace field of a Hecke group is the maximal real subfield of a cyclotomic field, so

$$\big[\,\mathbb{Q}(\lambda_q) : \mathbb{Q}\,\big] \;=\; \tfrac{1}{2}\phi(2q), \qquad \text{and} \qquad \big[\,\mathbb{Q}(\lambda_q) : \mathbb{Q}\,\big] \;=\; 2 \iff q \in \{4,5,6\}. \tag{1.4}$$

Since a surd manufactured on the traveling side is always the square root of an integer, a Hecke group other than $X(1)$ can share a field with that side only at those three orders. The list is closed by (1.4) and not by search. What (1.4) quantifies over is trace fields; it is not a statement about meetings at the level of the surd, which occur at every order, and Remark 2.12 exhibits one.

§3 — the level law and the descent criterion. Every integer trace $t \geq 3$ has a square root in a Fricke coset, at the level given by the squarefree part of $t+2$ (Proposition 3.1). On the arithmetically distinguished traces the level is determined by the discriminant alone: if D is a fundamental discriminant whose fundamental unit $\varepsilon_D = \frac{1}{2}(u + v\sqrt{D})$ has norm -1, and $t = \mathrm{Tr}(\varepsilon_D^2) = u^2+2$, then Theorem 3.3 gives

$$t+2 \;=\; u^2+4 \;=\; v^2 D, \qquad p \;=\; \mathrm{kernel}(D), \qquad \lambda_N \;=\; \varepsilon_D, \tag{1.5}$$

where kernel is the squarefree kernel and λ_N the eigenvalue on the floor. The same closed geodesic is thus recorded twice, on $\Gamma_0(p)^+$ with eigenvalue ε_D and on $X(1)$ with eigenvalue ε_D^2 (Corollary 3.5). Which classes at a given trace descend is settled exactly: a class descends if and only if the rung index n divides its content, necessary by a two-line identity and sufficient by a four-step construction (Theorem 3.6). The law is *level-relative*, and §3.4 says what that costs — the kernel level is the minimal admissible level, verified minimal over the range examined, and it is not the only one.

§4 — the single-inequality dichotomy. The first rung of $\Gamma_0(p)^+$ has trace $\sqrt{p}$ and the parabolic threshold is 2; comparing those two numbers decides eight distinct things, collected in §4.3 because no one of them is visible from any other. Among them: the floor's elliptic orders, its signature, and whether it is a Hecke group at all. Theorem 4.5 is the closure — $\Gamma_0(1)^+ = G(3)$, $\Gamma_0(2)^+ = G(4)$, $\Gamma_0(3)^+ = G(6)$, and the first floor with $\sqrt{p} > 2$ is not a triangle group, differing from $G(5)$ in area, torsion and abstract group type. Those three are exactly the finite entries of Takeuchi's list of arithmetic Hecke groups. The list $\{1, 2, 3\}$ of levels and the list $\{3, 4, 6\}$ of orders are one list read through $p = \lambda_q^2$ (Corollary 4.10), and the identification is Niven's theorem, which does four separate jobs in this paper (Remark 4.11).

§5 — systoles of the floors. Because $G(4)$ and $G(6)$ are Fricke floors, the grading (1.3) turns a systole from a search into a comparison of two explicit sequences, and Theorem 5.3 gives their systolic traces as $2\sqrt{2}$ and $2\sqrt{3}$ — quantified over the whole group. The two comparisons come out the same way for different reasons: at $p = 2$ the integer side is obstructed by a parity, at $p = 3$ by a quadratic non-residue. The third group has no such grading, and its minimum is obtained instead from the free-product normal form: the minimal hyperbolic trace of $G(5)$ is $2\varphi = 1 + \sqrt{5}$ (Theorem 5.5), whence the interval $(2, 2\varphi)$ contains no trace at all, so that $\sqrt{5}$ and 3 are not traces of $G(5)$ (Corollary 5.6). The three results stand on their own evidence and are not derived from one another. What they make visible is that the golden Hecke group and the golden floor are different surfaces carrying different systoles,

$$2 \operatorname{arccosh} \varphi \;=\; 2.122550124\ldots \qquad \text{against} \qquad 2 \ln \varphi \;=\; 2 \operatorname{Reg} \mathbb{Q}(\sqrt{5}) \;=\; 0.962423650\ldots, \tag{1.6}$$

the first on $G(5)$ and the second on $\Gamma_0(5)^+$ (Corollary 5.9): the golden Hecke group does not carry the golden regulator. It does carry $\frac{1}{2} \ln \varphi$ exactly, as the distance from the order-2 cone point to its own systolic axis (Proposition 5.13) — a distance, not a length in the spectrum, and Remark 5.14 keeps the two apart.

§6 — the doubling law and the bead ladder. At $D = 8$ both records of Corollary 3.5 can be written down and compared digit by digit: the length on $X(1)$ is $4 \ln(1 + \sqrt{2})$ and the length on the floor is $2 \ln(1 + \sqrt{2})$, exactly twice and exactly half (Proposition 6.1). The factor is not an accident of the example — the order of discriminant 32 carries no unit of norm -1, so the geodesic has no choice but to run to ε^2 (Proposition 6.2). The axis underneath carries more than its length. Theorem 6.8 shows that at $p = 2$ and $p = 5$ the stabiliser of the axis is infinite dihedral and its reflection set is exactly $\{\varepsilon^k : k \in \mathbb{Z}\}$, for all k: the cone points on the axis sit at the integer multiples of the regulator, indexed by the powers of the fundamental unit, and the parity of the index decides which of them the modular group sees (Theorem 6.5). The theorem has two hypotheses, and §6.4 gives an effective criterion deciding the first at an arbitrary floor; over the 28 fundamental discriminants $5 \leq D \leq 204$ with $N(\varepsilon_D) = -1$ it holds at exactly four, $D \in \{5, 8, 13, 53\}$.

§7 — reciprocity and threading. On $X(1)$ four descriptions name the same closed geodesics — passing through the orbit of the order-2 cone point, equalling one's own inverse, containing a symmetric matrix, being inverted by an involution with fixed point on the axis (Theorem 7.1) — and the passages number exactly two per period (Proposition 7.2). The existence question at a given trace then reduces to a single Diophantine criterion: some class of trace t threads if and only if

$$t^2 - 4 \;=\; s^2 + (2b)^2, \qquad b \neq 0, \tag{1.7}$$

is solvable in integers (Theorem 7.4). The two families of traces produced by §3 both satisfy (1.7), and neither is a competing mechanism: they are conditions under which the one criterion is already met. They do not exhaust it. Over $3 \le t \le 200$ there are 33 threading traces, of which 14 come from the first family and 4 from the second, while 15 — the largest of the three classes — lie outside both (Proposition 7.6). Reciprocity is strictly weaker than the norm condition on the fundamental unit, the two first coming apart at $t = 15$ (Proposition 7.8). The whole of it survives to $G(4)$ intact and fails at $G(5)$ in a way that is itself a result: the equivalence survives, the unit-level partner fails in both directions, and what co-tracks reciprocity there is archimedean — inner-reciprocal implies $|t^\sigma| > 2$ (Theorem 7.13), with the converse false.

§8 — the realization laws, and the trace gaps. Every integer is the trace of an element of the modular group and no other real number is: $\Theta(\mathrm{PSL}(2,\mathbb{Z})) = \mathbb{Z}$ (Theorem 8.1). At $G(4)$ the answer is again a closed form,

$$\Theta\big(G(4)\big) \;=\; 2\mathbb{Z} \,\cup\, \sqrt{2}\cdot\mathbb{Z}, \tag{1.8}$$

up to sign, proved from the exact matrix model (Theorem 8.2), so that the trace gaps of $G(4)$ in its ambient ring $\mathbb{Z}[\sqrt{2}]$ are exactly the mixed elements and the odd integers (Corollary 8.7). An absent value is absent outright rather than approached: the quantities belonging to the ambient group vary continuously along the trace line, and those belonging to the discrete group are supported on the realized set with no tail (Proposition 8.8). Asking instead which values are traces of *no* discrete group at all gives opposite answers on the two sides of the parabolic value. On the elliptic side the realized set is $\{2\cos(k\pi/m)\}$, hence countable with co-countable complement (Theorem 8.10, Corollary 8.11); on the hyperbolic side every $t > 2$ is realized, with the group exhibited (Theorem 8.12). The mechanism is one line: discreteness constrains rotation angles and does not constrain translation lengths.

§9 — the value identity and the mutual exclusion. At one of the three orders permitted by (1.4) the sharing is an equality of numbers and not merely of fields: $\lambda_5 = \varepsilon_5$, and the reason is classical — the fundamental unit of $\mathbb{Q}(\sqrt{5})$ is a cyclotomic unit at level 5, and λ_5 is that unit (Theorem 9.1). Theorem 9.5 then shows that no group relation can accompany the identity: at any Hecke group whose trace field is quadratic, "the parameter is a unit" and "the group is arithmetic" cannot both hold. By (1.4) the two groups other than $G(5)$ are not a sample but the rest of the population, so §9.4 completes an enumeration rather than tests a hypothesis. What the identity transports is nothing: the invariant trace fields differ, the groups share no finite-index subgroup, and no length, area, spectrum or cone datum crosses (Proposition 9.8). The theorem lives entirely inside $\mathbb{Q}(\sqrt{5})$ and touches neither group.

§10 — the parabolic trace, and finiteness. One trace value behaves unlike the rest: the parabolic classes of $\mathrm{PSL}(2,\mathbb{Z})$ are indexed bijectively by $n \in \mathbb{Z} \setminus \{0\}$, so a single trace carries infinitely many (Proposition 10.1). Under the primitivity filter the count is 2, and every trace value of $X(1)$ carries finitely many primitive classes (Corollary 10.3, Theorem 10.5). The infinity at the parabolic value is an infinity of repetition, not of distinct structure.

1.4 Method, and how the machine work is reported

Every computation reported below is exact wherever exactness is available: integer and quadratic-surd arithmetic on matrices, forms, traces and units, with floating point confined to the reporting of a decimal expansion or to a residual. Where a decimal appears in a load-bearing step, the exact value appears beside it. One convention governs every decimal printed here. A value written with a trailing ... is correct to the digits shown — either an exact truncation of the expansion or its correct rounding at the last printed digit, halves rounded up — and the ellipsis asserts only that the expansion continues, not that it continues as printed.

Machine work is *reported*, not performed in the prose. Each statement whose evidence is computational carries an italic *Verification.* paragraph stating what was checked and — inline, in the same sentence as the claim it qualifies — the population over which it was checked. Two kinds of population recur and are not interchangeable. A ball bounds word length and is a sample of the group; the free-product normal form bounds syllable depth and is exhaustive over conjugacy classes, which is why a syllable bound can support a census while a ball supports only a recall statement (§5.3). Where a bound is a recall bound the sentence says so, and says that the enumeration is silent outside it.

Independence is likewise reported rather than assumed. Several results below were obtained twice by computations sharing no step, and where that is so the second route is described so the reader can see what it does and does not duplicate; where two computations were made inside one project they are reported as corroboration and not as independent confirmation (Remark 2.5). Appendix A carries the instruments, the determinism convention under which each was run, the full population table, and the failures — including a control that passed a determinism check while computing the wrong quantity, which is the reason the appendix distinguishes reproducibility from correctness. The catalogue of open questions in §11 is likewise drawn from a standing index of such questions maintained alongside the work throughout, rather than assembled at the end from what the results happened to leave over.

1.5 Notation and conventions

Traces are taken up to sign throughout (the census counts reported in §5 and Appendix A are the one exception: they are sign-distinguished), an element of $\mathrm{PSL}(2,\mathbb{R})$ having a well-defined trace only up to sign, and $\Theta(G)$ denotes the set of traces realized by a group G. For a hyperbolic element of trace t the translation length of its axis is $2\operatorname{arccosh}(|t|/2)$, strictly increasing in $|t|$, so trace statements and length statements determine each other and each is made at whichever of the two is established. Fundamental units are indexed by field discriminant, ε_D denoting the fundamental unit of $\mathbb{Q}(\sqrt{D})$, and Reg its regulator. To a matrix of determinant 1 and trace t is attached the fixed-point form of discriminant t^2-4, and the *content* of a class of forms is the gcd of the coefficients of any of its forms (§3.1).

Two symbols that would otherwise collide. φ denotes the golden ratio $\frac{1}{2}(1+\sqrt{5})$ throughout, and ϕ denotes Euler's totient — as in (1.4), where $\phi(2q)$ is the totient and φ nowhere appears. Both quantities are unavoidable in this paper and the distinction between the two glyphs is load-bearing wherever they occur together, in §2 and §9. Relatedly, Φ_q denotes the q-th cyclotomic polynomial (§9), and $\gamma(m)$ rather than $G(m)$ denotes the fixed-point gap $2\sin(\pi/m)$ of §2, the letter G being reserved for the Hecke groups.

Numbering. Each section runs two independent counters: statements are numbered Theorem $N.k$, Proposition $N.k$, and so on, consecutively through the section, while displayed equations carry their own consecutive counter $(N.1), (N.2), \ldots$. Every display is numbered and there are no unnumbered displays. The two counters collide as numerals — equation (3.3) is not Theorem 3.3 — so the environment is always named for a statement (“by Theorem 3.6”, “Proposition 3.12(ii)”) and never for a display (“by (3.29)”). Arguments reference displays by number rather than re-deriving them inline.

Terms this paper defines for its own use. Each is introduced where it is first needed and is listed here so that none is mistaken for standard terminology: *seat* and the *torsion* / *traveling* classification of occurrences (§2.1); *floor* for $\Gamma_0(p)^+$ in the role it plays there, *rung* for a coset element of trace $k\sqrt{Q}$, and *descends from* (§3.1); *bead*, *necklace*, *visible* and *invisible* beads, and *pivot* (§6.2; in §7 the same word names the inverting involution whose fixed point the §6 usage picks out — context disambiguates); *rhyme* and *edge* for value-coincidences without and with a forcing mechanism (§2.4); *realized trace set* and *trace gap* (§8.1); *universally unrealized* (§8.6). Everything else is standard.

2. The atlas of surd occurrences, and the organizing theorem

A quadratic surd enters this neighborhood by two structurally unrelated routes. It can be manufactured by a torsion element — as the gap between an elliptic trace and the value 2, or as a product of sines taken over a cone point's order — and it can be manufactured by a hyperbolic class, as the eigenvalue gap of an integer trace or as the discriminant of the field that trace generates. The same number arrives by both routes often enough that a catalogue indexed by *value* would be worthless. This section builds the catalogue indexed by *occurrence* instead, names the manufacturing mechanism at every entry, and then closes it: the Hecke groups whose trace field is quadratic are exactly $G(4)$, $G(5)$, $G(6)$, because $[\mathbb{Q}(\lambda_q) : \mathbb{Q}] = \phi(2q)/2$. The neighborhood's right-hand edge is a theorem and not a survey — and §2.5 states exactly what that completeness quantifies over, and what it does not.

2.1 Seats, and the mechanisms that manufacture surds

Throughout, $G(q)$ denotes the Hecke group generated by $z \mapsto -1/z$ and $z \mapsto z + \lambda_q$, where

$$\lambda_q \;=\; 2\cos(\pi/q), \qquad q \geq 3 \tag{2.1}$$

[2], and $G(3) = \mathrm{PSL}(2,\mathbb{Z}) = X(1)$.

One term is this paper's own and is used only for brevity. A **seat** is a triple — a group of the neighborhood, a conjugacy class or cone point of that group, and its trace — and a surd *sits at* a seat. The organizing convention of the catalogue is that the classification attaches to the seat and never to the number: the same surd may sit at two seats of different classes, and the atlas is laid out so that this is visible rather than hidden. We call an occurrence **torsion** when the mechanism producing it runs on the order of a cone point, and **traveling** when it runs on the trace of a hyperbolic class or on the discriminant of a field.

The torsion mechanisms. There are two of them, and they are different functions of the order. The first is the fixed-point gap at the order-m elliptic trace λ_m,

$$\gamma(m) \;=\; \sqrt{4-\lambda_m^2} \;=\; 2\sin(\pi/m); \tag{2.2}$$

the second is the cone-point half-product,

$$H(m) \;=\; \prod_{k=1}^{\lceil m/2\rceil-1} 2\sin(k\pi/m). \tag{2.3}$$

We label them **M-T1** and **M-T2**. (The gap is written γ rather than G because G is the Hecke group throughout this paper; the two are distinct functions and the notation must keep them apart.)

The traveling mechanisms. There are three. **M-V1** is the eigenvalue gap $\sqrt{t^2-4}$ at a hyperbolic class of integer trace t, which factors through the conductor f and the fundamental discriminant D_0 of the field the class generates,

$$\sqrt{t^2 - 4} \;=\; f\sqrt{D_0}; \tag{2.4}$$

M-V2 is the field discriminant $\sqrt{D_0}$ itself, which is the surd entering the class number formula

$$L(1, \chi_D) \;=\; \frac{2\log \varepsilon_D}{\sqrt{D}} \tag{2.5}$$

[3]; and **M-V3** is the Lagrange–Hurwitz extremum of the Markov spectrum, reached through an indefinite form with infinite automorph group [4].

Proposition 2.1 (the half-product is the square root of an integer). *For every integer $m \geq 2$, the classical product identity*

$$\prod_{k=1}^{m-1} 2\sin(k\pi/m) \;=\; m \tag{2.6}$$

folds to

$$H(m) \;=\; \sqrt{m} \quad (m \text{ odd}), \qquad H(m) \;=\; \sqrt{m/2} \quad (m \text{ even}). \tag{2.7}$$

In particular $H(m)$ is the square root of an integer at every order, so $\deg_{\mathbb{Q}} H(m) \leq 2$ for every m.

Proof. The factors of (2.6) satisfy $2\sin(k\pi/m) = 2\sin((m-k)\pi/m)$, so the indices $k = 1, \dots, m-1$ pair off under $k \mapsto m-k$. For m odd no index is self-paired and the pairs number $(m-1)/2 = \lceil m/2 \rceil - 1$, so (2.6) reads $m = H(m)^2$. For m even the index $k = m/2$ is self-paired and contributes $2\sin(\pi/2) = 2$, leaving $m/2 - 1 = \lceil m/2 \rceil - 1$ pairs, so (2.6) reads $m = 2H(m)^2$. ■

Verification. Both halves were checked at every order $2 \leq m \leq 12$ — eleven orders — at 50 decimal places with tolerance 10^{-40}: the full product (2.6) returned m in 11 of 11 cases, and the closed form (2.7) in 11 of 11 [App. A].

Proposition 2.1 is what makes M-T2 structurally different from M-T1. A traveling surd is a square root of an integer by construction — this is immediate from (2.4) and from M-V2 — and so has degree at most 2 over $\mathbb{Q}$. M-T2's output has that same shape at *every* order, and M-T1's does not.

Proposition 2.2 (where the two torsion mechanisms agree). *Over the range $2 \leq m \leq 12$, $\gamma(m) = H(m)$ at $m = 3$ and at $m = 4$ and at no other order:*

$$\gamma(3) \;=\; H(3) \;=\; \sqrt{3}, \qquad \gamma(4) \;=\; H(4) \;=\; \sqrt{2}. \tag{2.8}$$

The degrees of the two outputs part company immediately after. At $m = 2, \dots, 12$,

$$\deg_{\mathbb{Q}} \gamma(m) \;=\; 1,\ 2,\ 2,\ 4,\ 1,\ 6,\ 4,\ 6,\ 2,\ 10,\ 4, \tag{2.9}$$

against $\deg_{\mathbb{Q}} H(m) \leq 2$ throughout; at $m = 5$, for instance,

$$\gamma(5) \;=\; 2\sin(\pi/5) \;=\; 1.1755705\ldots \text{ (degree 4)}, \qquad H(5) \;=\; \sqrt{5} \text{ (degree 2)}. \tag{2.10}$$

The torsion $\sqrt{5}$ of §2.3 is therefore H and not γ, and the range $2 \le m \le 12$ travels with (2.8): the agreement at $m = 3$ and $m = 4$ is asserted inside that range and not beyond it.

Remark 2.3 (order 4 is the unique triple coincidence). At order 4, and at no other order whatever, the defining trace, the fixed-point gap and the half-product are one number:

$$2\cos(\pi/4) \;=\; 2\sin(\pi/4) \;=\; \sqrt{2}, \tag{2.11}$$

because $\pi/4$ is the fixed point of $\theta \mapsto \pi/2-\theta$, the map exchanging cos and sin. Uniqueness here is unbounded, since $2\cos(\pi/m) = 2\sin(\pi/m)$ already forces $m = 4$. Four separate checks of the coincidence returned true, 4 of 4 [App. A].

Proposition 2.4 (the conductor separates M-V1 from M-V2). *By (2.4) the eigenvalue gap and the field discriminant agree exactly when the conductor $f = 1$, and differ whenever $f > 1$. Four seats on $X(1)$'s own trace line exhibit the separation.*

Table 2.1. *The seats of $X(1)$ with $f > 1$ in the range $t \le 12$.*

t	t^2-4	D_0	f	M-V1 returns	M-V2 returns
6	32	8	2	$\sqrt{32} = 4\sqrt{2}$	$\sqrt{8} = 2\sqrt{2}$
7	45	5	3	$\sqrt{45} = 3\sqrt{5}$	$\sqrt{5}$
10	96	24	2	$\sqrt{96}$	$\sqrt{24}$
11	117	13	3	$\sqrt{117} = 3\sqrt{13}$	$\sqrt{13}$

2.2 The trace line of $X(1)$

The traveling side's data on the modular surface is a single table. Write $D = t^2-4$ for the discriminant at trace t, D_0 for the fundamental discriminant and f for the conductor, so that

$$D \;=\; t^2-4 \;=\; f^2 D_0, \tag{2.12}$$

and $h^+(D)$ for the narrow class number of forms of discriminant D [5, 6]. The last column records whether some class of that trace is S-reciprocal in the sense of §7.

Table 2.2. *The trace line of $X(1)$, machine-computed over $0 \le t \le 12$; the range is the table's own and nothing outside it is claimed. The traces $t = 0, 1$ are elliptic, $t = 2$ is parabolic, and every $t \ge 3$ is hyperbolic.*

t	D	D_0	f	field	$h^+(D)$	oriented	ε	$N(\varepsilon)$	S-rec.
0	-4	-4	1	$\mathbb{Q}(i)$	1	1	finite	—	no
1	-3	-3	1	$\mathbb{Q}(\omega)$	1	1	finite	—	no
2	0	—	—	—	—	∞	—	—	—
3	5	5	1	$\mathbb{Q}(\sqrt{5})$	1	1	φ	-1	yes

t	D	D_0	f	field	$h^+(D)$	oriented	ε	$N(\varepsilon)$	S-rec.
4	12	12	1	$\mathbb{Q}(\sqrt{3})$	2	2	$2+\sqrt{3}$	$+1$	no
5	21	21	1	$\mathbb{Q}(\sqrt{21})$	2	2	$(5+\sqrt{21})/2$	$+1$	no
6	32	8	2	$\mathbb{Q}(\sqrt{2})$	2	3	$1+\sqrt{2}$	-1	yes
7	45	5	3	$\mathbb{Q}(\sqrt{5})$	2	3	φ	-1	yes
8	60	60	1	$\mathbb{Q}(\sqrt{15})$	4	4	$4+\sqrt{15}$	$+1$	no
9	77	77	1	$\mathbb{Q}(\sqrt{77})$	2	2	$(9+\sqrt{77})/2$	$+1$	no
10	96	24	2	$\mathbb{Q}(\sqrt{6})$	4	6	$5+2\sqrt{6}$	$+1$	no
11	117	13	3	$\mathbb{Q}(\sqrt{13})$	2	3	$(3+\sqrt{13})/2$	-1	yes
12	140	140	1	$\mathbb{Q}(\sqrt{35})$	4	4	$6+\sqrt{35}$	$+1$	no

The four S-reciprocal traces of the table carry the symmetric representatives

$$\begin{pmatrix}1&1\\1&2\end{pmatrix},\quad \begin{pmatrix}1&2\\2&5\end{pmatrix},\quad \begin{pmatrix}2&3\\3&5\end{pmatrix},\quad \begin{pmatrix}1&3\\3&10\end{pmatrix} \tag{2.13}$$

at $t = 3, 6, 7, 11$ respectively. Where $f > 1$ with f prime the oriented class count exceeds $h^+(D)$ and the excess is $h^+(D_0)$ (in general the count is $\sum_{d|f} h^+(d^2 D_0)$); over Table 2.1's range: at $t = 6$ the count is $3 = h^+(32) + h^+(8) = 2 + 1$, at $t = 7$ it is $3 = h^+(45) + h^+(5) = 2 + 1$, at $t = 10$ it is $6 = h^+(96) + h^+(24) = 4 + 2$, and at $t = 11$ it is $3 = h^+(117) + h^+(13) = 2 + 1$. Every narrow class number in the table was recomputed by a reduced-form-cycle routine written independently of the routine that produced it, against a five-value control fixed before that routine existed; all ten hyperbolic rows were checked, 32 checks, 32 pass, 0 fail [App. A].

> **Remark 2.5 (the imprimitive third class at $t = 6$).** The oriented class counts $1, 2, 2, 3$ at $t = 3, 4, 5, 6$ were obtained here from cycles of reduced indefinite forms, and they reproduce exactly a count obtained from an enumeration of reduced words in the generators — a different route to the same four numbers. The decomposition
>
> $$3 \;=\; h^+(32) + h^+(8) \;=\; 2 + 1 \tag{2.14}$$
>
> identifies the third class at $t = 6$ as the imprimitive one, sitting at the non-fundamental discriminant 32 over the fundamental 8. Both computations were carried out inside the same project; the agreement is corroboration, and is reported as such rather than as independent confirmation.

2.3 The seat index

The tables below list, for each of the three surds around which the neighborhood is organized, every occurrence for which a mechanism is named. The index is complete for the mechanisms named and for no others: a seat whose mechanism is not established has no row, by design. The source catalogue also carries occurrences drawn from outside the modular and Hecke groups; those lie outside this paper's domain and are not tabulated here, so the row counts below are smaller than the catalogue's own — two entries fewer for $\sqrt{2}$ and four fewer for $\sqrt{5}$.

Table 2.3. *The $\sqrt{2}$ seats.*

	seat	mechanism	class
R2-a	$G(4)$'s order-4 cone point, as the fixed-point gap $2\sin(\pi/4)$	M-T1	torsion
R2-b	$G(4)$'s order-4 cone point, as the half-product $H(4) = \sqrt{4/2}$	M-T2	torsion
R2-c	$G(4)$'s defining trace $\lambda_4 = 2\cos(\pi/4)$	the order-4 rotation's trace	torsion
R2-d	$X(1)$ at $t = 6$, the disc-8 class: $M = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, anchors $1 \pm \sqrt{2}$, $\ell = 4\log(1+\sqrt{2})$	M-V1 / M-V2 at $f = 2$	traveling
R2-e	the fixed point of the involution $t \mapsto \sqrt{4-t^2}$	(2.11)	torsion, self-dual

> **Proposition 2.6 (the $\sqrt{2}$ torsion seat is quadruply determined).** *R2-a, R2-b, R2-c and R2-e are one point — $G(4)$'s order-4 cone point — reached by four descriptions that coincide only there, the coincidence being (2.11). No other surd in the atlas has a torsion seat determined four times over.*

R2-d is a genuinely different object, and it is different in *class*, not in surface: it is $X(1)$'s own $t = 6$ disc-8 closed geodesic, of length $4\log(1+\sqrt{2}) = 3.52549\ldots$, whose surd carries the conductor $f = 2$, and the surfaces $X(1)$ and $G(4)$ are commensurable. The pairing R2-{a,b,c,e} ↔ R2-d is the catalogue's entry R-3, and its classification is taken up in §2.4.

Table 2.4. *The $\sqrt{3}$ seats.*

	seat	mechanism	class
R3-a	$X(1)$'s order-3 cone point ρ, where gap and half-product agree, $\gamma(3) = H(3) = \sqrt{3}$	M-T1 = M-T2	torsion

	seat	mechanism	class
R3-b	$G(6)$'s order-6 cone point, $H(6) = \sqrt{6/2} = \sqrt{3}$	M-T2	torsion
R3-c	$G(6)$'s defining trace $\lambda_6 = 2\cos(\pi/6) = \sqrt{3}$	the order-6 rotation's trace	torsion
R3-d	$X(1)$ at $t = 4$, disc 12: the geodesic through ρ, $\ell = 2\log(2+\sqrt{3})$, $\varepsilon = 2+\sqrt{3}$, $N(\varepsilon) = +1$	M-V1 = M-V2 at $f = 1$	traveling
R3-e	the elliptic weight $1/(3\sqrt{3}) = L(1, \chi_{-3})/\pi$	the elliptic scalar of the trace formula at $m = 3$	torsion

Table 2.5. *The* $\sqrt{5}$ *seats.*

	seat	mechanism	class
R5-a	$G(5)$'s order-5 cone point, $H(5) = 2\sin(\pi/5) \cdot 2\sin(2\pi/5) = \sqrt{5}$	M-T2, with no arithmetic input whatever	torsion
R5-b	$G(10)$'s order-10 cone point, $H(10) = \sqrt{10/2} = \sqrt{5}$	M-T2, through (2.15)	torsion
R5-c	$X(1)$ at $t = 3$, the systole: $\sqrt{t^2-4} = \varphi^2 - \varphi^{-2} = \sqrt{5}$, $\ell = 4\log\varphi$	M-V1 = M-V2 at $f = 1$	traveling
R5-d	the field discriminant of $\mathbb{Q}(\sqrt{5})$, through $L(1, \chi_5) = 2\log\varepsilon/\sqrt{5}$	M-V2	traveling
R5-e	the Lagrange–Hurwitz spectrum minimum	M-V3	traveling

Proposition 2.7 (two forced identities inside the torsion side). *For odd n,*

$$H(2n) \;=\; \sqrt{2n/2} \;=\; \sqrt{n} \;=\; H(n), \tag{2.15}$$

*by (2.7). Hence R3-a ↔ R3-b and R5-a ↔ R5-b are identities and not coincidences: the order-*6 *and order-*3 *cone points are forced to manufacture the same surd, as are the order-*10 *and order-*5 *cone points.*

R5-b is recorded as a declared extension of the catalogue. $G(10)$ is a Hecke group but not one of the three the neighborhood names; the seat is reached from inside the catalogue by (2.15) and is marked so that it can be struck if the reach is judged too long. Nothing else in the catalogue depends on it.

2.4 Edges and rhymes

Two occurrences are joined by an **edge** when an established derivation connects them; the derivation is shown, or the edge is not drawn. They are joined by a **rhyme** when they share a value and no mechanism connects them. The distinction is the catalogue's whole point, and the posture attached to it is that rhymes are collected and never asserted as evidence.

Table 2.6. *The mechanism-edges.*

	edge	the derivation
E-1	every M-T1 and every M-V1 occurrence, on any group, is $\sqrt{\lvert t^2-4\rvert}$ evaluated at a different trace	one function, evaluated on the two sides of the surd-blind value $t = 2$
E-2	every M-T2 occurrence is (2.6) at a different order	one classical identity
E-3	the doubling edge, $H(2n) = H(n)$ for odd n	(2.15); instantiated at $\sqrt{3}$ (orders 3, 6) and $\sqrt{5}$ (orders 5, 10)
E-4	the conductor edge, M-V1 $= f\cdot$ M-V2	(2.4)
E-5	λ_q generates $\mathbb{Q}(\zeta_{2q})^+$, which for $q = 4, 5, 6$ is the real quadratic field of discriminant 8, 5, 12 — and each of those discriminants is realized on $X(1)$'s trace line, at $t = 6$, $t = 3$, $t = 4$	conductor–discriminant for abelian fields, together with the scan of Table 2.2
E-6	the cyclotomic-unit edge, $\lambda_5 = 2\cos(\pi/5) = -(\zeta_5^2 + \zeta_5^3) = \varphi = \varepsilon_5$	the cyclotomic units of a real abelian field of class number 1; at $q = 5$ only
E-7	the reciprocity edge, S-reciprocity at trace $t \iff t^2 - 4$ is a sum of two squares	derived in §7
E-8	the duality edge, $t \mapsto \sqrt{4-t^2}$ is an involution of the elliptic range pairing $0 \leftrightarrow 2$ and $1 \leftrightarrow \sqrt{3}$, with unique fixed point $\sqrt{2}$	(2.11)
E-9	the branch-point edge: the torsion angle and the traveling length are one analytic function, $2\arccos(t/2) = -i\cdot 2\,\mathrm{arccosh}(t/2)$ continued past $t = 2$	$\arccos x = -i\,\mathrm{arccosh}\,x$ across the branch point

Table 2.7. *The rhymes.*

	rhyme	the two mechanisms	status
R-1	$\sqrt{5}$: $G(5)$'s cone point $\leftrightarrow X(1)$'s systole	M-T2, a pentagon with no arithmetic in it, against M-V1 = M-V2 at disc 5	rhyme; the catalogue's founding instance
R-2	$\sqrt{3}$: $X(1)$'s cone point $\leftrightarrow$ the disc-12 geodesic in $\mathbb{Q}(\sqrt{3})$	M-T1 = M-T2 against M-V1	rhyme
R-3	$\sqrt{2}$: $G(4)$'s cone point $\leftrightarrow X(1)$'s $t = 6$ disc-8 class	M-T1 = M-T2 = trace against M-V1 at $f = 2$	edge, by Proposition 2.8 (reclassified from the catalogue's sealed rhyme)
R-4	φ: λ_5 as a torsion trace $\leftrightarrow \varphi$ as $X(1)$'s systolic axis anchor	the order-5 rotation's trace against a boundary fixed point	adjudicated in §9
R-5	$2\sqrt{\varphi} \leftrightarrow \sqrt{5}$, each the systolic surd of its own group	M-V1 on $G(5)$ against M-V1 on $X(1)$	neither a rhyme nor an edge — same mechanism, different values; recorded to prevent a false pairing
R-7	$5^{1/4}$ against anything	a composite mechanism against —	isolated: no rhyme and no edge

The catalogue records nine mechanism-edges and, under its own counting convention — which omits the §9-adjudicated R-4 and the declined R-5 — five rhymes; one of those five is an occurrence outside this paper's domain and is not tabulated above. Both counts are bookkeeping over the present state of the catalogue and are not offered as evidence of anything. They move whenever a mechanism is found for a rhyme, which is exactly what the next proposition does to R-3.

Proposition 2.8 (the two $\sqrt{2}$ seats are consecutive rungs of one progression). $G(4) = \Gamma_0(2)^+$*, and the Fricke coset of* $\Gamma_0(2)^+$ *has trace set* $\sqrt{2} \cdot \mathbb{Z}$ *[7]. Its first rung has trace* $\sqrt{2} = 2\cos(\pi/4)$ *and is elliptic of order* 4 *— the cone point carrying R2-a, R2-b and R2-c. Its second rung N has trace* $2\sqrt{2}$ *and is hyperbolic, with*

$$\mathrm{tr}(N^2) \;=\; (2\sqrt{2})^2 - 2 \;=\; 6, \qquad N^2 \in \Gamma_0(2) \subset \mathrm{PSL}(2, \mathbb{Z}), \tag{2.16}$$

and

$$\begin{aligned} \ell(N) \;&=\; 2\,\mathrm{arccosh}\,\sqrt{2} \;=\; 2\log(1 + \sqrt{2}) \;=\; 1.762747\ldots, \\ \ell(N^2) \;&=\; 4\log(1 + \sqrt{2}) \;=\; 3.525494\ldots, \end{aligned} \tag{2.17}$$

the second of which is R2-d's length. The two seats of R-3 are therefore consecutive rungs of one arithmetic progression inside one group, separated by nothing but the value $t = 2$ *— which is E-1's surd-blind wall.*

Verification. 10 checks, 10 pass, 0 fail [App. A].

R-3, reclassified. The catalogue as sealed records R-3 as a rhyme; Proposition 2.8 supplies the grounds on which it is here reclassified as an edge. Under the rhyme reading R-3 would remain a shared value whose mechanisms — torsion at the cone point, traveling at conductor $f = 2$ — are unmatched, and the entry stays collected and unasserted; under the adopted edge reading, Proposition 2.8 supplies precisely the *same mechanism, forced identity* that makes R3-a ↔ R3-b and R5-a ↔ R5-b edges by Proposition 2.7, and the entry moves. Nothing in Proposition 2.8 depends on which reading is adopted.

> **Proposition 2.9 (the same test on the other two rhymes: a trichotomy).** *Applying the progression test of Proposition 2.8 to R-2 and R-1 returns opposite answers, and for different reasons.*
>
> | rhyme | the test | outcome |
> |---|---|---|
> | R-3, $\sqrt{2}$ | both seats are rungs of the progression $\sqrt{2} \cdot \mathbb{Z}$ in $\Gamma_0(2)^+$ | reclassified an edge (Proposition 2.8) |
> | R-2, $\sqrt{3}$ | $\Gamma_0(3)^+ = G(6)$ does carry a progression $\sqrt{3} \cdot \mathbb{Z}$ whose first rung $\sqrt{3}$ is the order-6 cone point; but the disc-12 traveling seat is $X(1)$ at the integer trace $t = 4$, in the other coset. Different progressions | stays a rhyme |
> | R-1, $\sqrt{5}$ | $\Gamma_0(5)^+$ is not a Hecke group at all, since $\sqrt{5} > 2$; so $G(5)$ carries no Fricke-coset progression, and there is no common ladder of which φ and the traveling $\sqrt{5}$ could be rungs | stays a rhyme |
>
> *The founding instance therefore survives for a reason rather than by default: R-1 is a rhyme because $\sqrt{5} > 2$.*

Proposition 2.9 carries the same proposal status as Proposition 2.8. Its R-2 and R-1 rows agree with the sealed classifications and supply their mechanism; only the R-3 row would move a classification, and that row is the one left open above.

2.5 The organizing theorem

The two families do not meet through the value $t = 2$: the surd $\sqrt{|t^2 - 4|}$ vanishes there, and E-1's one function is blind at exactly that point. Where they meet is in the field, and the meeting is completely determined.

Theorem 2.10 (the organizing theorem). *For every integer $q \geq 3$,*

$$\mathbb{Q}(\lambda_q) \;=\; \mathbb{Q}(\zeta_{2q})^+, \tag{2.18}$$

so that

$$[\,\mathbb{Q}(\lambda_q) : \mathbb{Q}\,] \;=\; \tfrac{1}{2}\phi(2q), \tag{2.19}$$

and

$$[\,\mathbb{Q}(\lambda_q) : \mathbb{Q}\,] \;=\; 2 \iff q \in \{4,5,6\}. \tag{2.20}$$

Since every surd manufactured by M-V1 or M-V2 is the square root of an integer, and so lies in a real quadratic field, a Hecke group other than $X(1)$ can share a trace field with the traveling family only for $q \in \{4,5,6\}$. The list is closed by (2.20), not by search.

The scope of "only three" is exactly this and no more: (2.20) quantifies over the trace fields $\mathbb{Q}(\lambda_q)$. It does not quantify over surd-level meetings between the two families, which occur at every order — see Remark 2.12.

Proof.

Step 1 (the field and its degree). $\lambda_q = \zeta_{2q} + \zeta_{2q}^{-1}$, so $\mathbb{Q}(\lambda_q)$ is the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{2q})$, which is (2.18); it has index 2 in $\mathbb{Q}(\zeta_{2q})$ [8], and $[\mathbb{Q}(\zeta_{2q}) : \mathbb{Q}] = \phi(2q)$, giving (2.19).

Step 2 (the fibre over the value 4). By (2.19), the degree equals 2 precisely when $\phi(2q) = 4$, and

$$\phi(2q) \;=\; 4 \iff 2q \in \{5,\ 8,\ 10,\ 12\}. \tag{2.21}$$

The argument $2q$ is even, so $2q \in \{8, 10, 12\}$, i.e. $q \in \{4,5,6\}$ — which is (2.20).

Step 3 (the rest of the range). At $q = 3$, (2.19) gives $\phi(6)/2 = 1$: the trace field is $\mathbb{Q}$ itself, and the group is $X(1)$, which is the traveling family's own home rather than a further meeting. For $q \geq 7$ the degree is at least 3. ■

Verification. The degree (2.19) was recomputed at every q with $3 \leq q \leq 200$ — exhaustive over that range — and returned $\{4,5,6\}$ and nothing else, 0 failures [App. A]. The degrees at $q = 3, \dots, 12$ are

$$1,\ 2,\ 2,\ 2,\ 3,\ 4,\ 3,\ 4,\ 5,\ 4. \tag{2.22}$$

Proposition 2.11 (the three meetings, in full). *Each of the three quadratic trace fields permitted by Theorem 2.10 is realized on $X(1)$'s trace line, and at exactly one of the three the torsion parameter is itself the fundamental unit of the traveling side.*

Table 2.8. *The three meetings.*

group	λ_q	trace field	field disc.	$X(1)$ traces realizing that discriminant (traces $t \le 40$)	$\lambda_q = \varepsilon$?
$G(4)$	$\sqrt{2}$	$\mathbb{Q}(\sqrt{2})$	8	$t = 6$ $(f = 2)$, $t = 34$ $(f = 12)$	no: $\varepsilon_8 = 1 + \sqrt{2}$, and $N(\lambda_4) = -2$
$G(5)$	φ	$\mathbb{Q}(\sqrt{5})$	5	$t = 3$ $(f = 1)$, $t = 7$ $(f = 3)$, $t = 18$ $(f = 8)$	yes: $\varepsilon_5 = \varphi$, and $N(\lambda_5) = -1$
$G(6)$	$\sqrt{3}$	$\mathbb{Q}(\sqrt{3})$	12	$t = 4$ $(f = 1)$, $t = 14$ $(f = 4)$	no: $\varepsilon_{12} = 2+\sqrt{3}$, and $N(\lambda_6) = -3$

The three norms are cross-checked against the cyclotomic polynomials by $|N(\lambda_q)| = |\Phi_q(-1)|$, *which at* $q = 3, \dots, 8$ *gives*

$$1,\ 2,\ 1,\ 3,\ 1,\ 2, \tag{2.23}$$

matching the table exactly.

The unit at $q = 5$ is the one place where the meeting is tighter than a shared field, and §9 takes up what that does and does not license.

Remark 2.12 (what "exactly three" does not quantify over). By Proposition 2.1, M-T2 returns the square root of an integer at *every* order. Meetings between the torsion and traveling families at the level of the surd are therefore not merely possible but present throughout the torsion family, and the catalogue exhibits one in Table 2.5: $G(10)$, with

$$H(10) \;=\; \sqrt{10/2} \;=\; \sqrt{5} \;=\; 2.2360679774997896964\dots \tag{2.24}$$

recomputed to 40 decimal places, is a fourth Hecke group sharing $\mathbb{Q}(\sqrt{5})$ with the traveling family. What Theorem 2.10 closes is the list of Hecke groups whose *trace field* is quadratic. Any reading of it as a statement about field-meetings in general is refuted by (2.24) and by (2.7).

3. The squaring correspondence and the level law

A single closed geodesic can carry two records. In the modular group it appears as a conjugacy class of integer trace t; in a Fricke group it appears one step earlier, as a class of trace $\sqrt{t+2}$ whose square is the first. This section shows that the second record is not an accident of the first. The square root exists for every integer trace $t \geq 3$; but on the arithmetically distinguished traces $t = u^2 + 2$ the level at which it lives is determined by the discriminant alone, the eigenvalue there is the fundamental unit itself, and the classes that descend are singled out by a divisibility condition on their content which is necessary and sufficient. The law so obtained is *level-relative*, and §3.4 states exactly what that qualifier costs and what it leaves intact.

3.1 Notation, and the construction

Let $p \geq 1$ be a squarefree integer, $\Gamma_0(p)$ the Hecke congruence subgroup of level p, and

$$w_p \;=\; \frac{1}{\sqrt{p}}\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \tag{3.1}$$

the Fricke involution [7], so that

$$\Gamma_0(p)^+ \;=\; \langle\, \Gamma_0(p),\, w_p \,\rangle. \tag{3.2}$$

We call $\Gamma_0(p)\, w_p$ the *Fricke coset*. More generally, at an exact divisor $Q \,\|\, N$ the Atkin–Lehner coset of $\Gamma_0(N)$ has trace set $\sqrt{Q}\cdot\mathbb{Z}$ [7]; the coset element of trace $k\sqrt{Q}$ will be called the *k-th rung*. Two further conventions are this paper's own and are used only for brevity: we call $\Gamma_0(p)^+$, in the role it plays below, the *floor* at level p, and we say that a conjugacy class of $\mathrm{PSL}(2,\mathbb{Z})$ *descends from* $\Gamma_0(p)^+$ when it contains N^2 for some N in the Fricke coset.

To a matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of determinant 1 and trace t we attach, as usual, the binary quadratic form

$$\big(C,\ D-A,\ -B\big) \tag{3.3}$$

of discriminant $t^2 - 4$, the fixed-point form of M; conjugacy classes of hyperbolic elements correspond to classes of such forms, and the *content* of a class is the gcd of the coefficients of any of its forms.

Proposition 3.1. *Let* $t \geq 3$ *be an integer and write*

$$t + 2 \;=\; p\,n^2, \qquad p \text{ squarefree}, \quad n \geq 1; \tag{3.4}$$

this factorisation is unique. Put

$$N \;=\; T^{\,n} w_p \;=\; \frac{1}{\sqrt{p}}\begin{pmatrix} np & -1 \\ p & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

Then N lies in the Fricke coset of $\Gamma_0(p)^+$ *and* $\operatorname{tr} N = n\sqrt{p}$*, while*

$$N^2 \;=\; \begin{pmatrix} n^2 p - 1 & -n \\ np & -1 \end{pmatrix} \tag{3.6}$$

has determinant 1*, trace* $n^2 p - 2 = t$*, and lower-left entry divisible by* p*; hence* $N^2 \in \Gamma_0(p) \subset \mathrm{PSL}(2,\mathbb{Z})$.

Proof. Both (3.5) and (3.6) are direct computations; the determinant, the trace and the congruence on the lower-left entry are read off (3.6). ■

Verification. The construction was carried out in exact integer arithmetic for every integer trace t with $3 \le t \le 80$ — 78 traces, 0 failures — each instance checked against its own invariants (det = 1, tr = t, lower-left $\equiv 0 \bmod p$) [App. A].

Remark 3.2 (the degenerate branch). The case $p = 1$ — equivalently, $t + 2$ a perfect square — is not an exception bolted onto Proposition 3.1 but the same formula at $p = 1$:

$$\Gamma_0(1)^+ \;=\; \mathrm{PSL}(2,\mathbb{Z}), \qquad w_1 \;=\; \mathrm{id}, \tag{3.7}$$

so the square root is already in the modular group, and the trace-t class is imprimitive.

Proposition 3.1 is uniform and purely geometric: *every* integer trace $t \ge 3$ has a Fricke square root, at the level p read off the squarefree part of $t + 2$. What is selective is the arithmetic that can sit underneath it, and that is the subject of the rest of the section.

3.2 The level law

Write kernel(m) for the squarefree kernel of a positive integer m, that is, the unique squarefree k with

$$m \;=\; k \cdot (\text{square}). \tag{3.8}$$

Theorem 3.3 (the level law). *Let D be a fundamental discriminant whose fundamental unit*

$$\varepsilon_D \;=\; \frac{u + v\sqrt{D}}{2} \tag{3.9}$$

has norm -1*, and put*

$$t \;=\; \mathrm{Tr}(\varepsilon_D^2) \;=\; u^2 + 2. \tag{3.10}$$

Then, in the notation of Proposition 3.1, the level-selection identity is

$$t + 2 \;=\; u^2 + 4 \;=\; v^2 D, \tag{3.11}$$

the level is

$$p \;=\; \mathrm{kernel}(D), \tag{3.12}$$

the rung is

$$n \;=\; v\sqrt{D/p}, \tag{3.13}$$

and the eigenvalue on the floor is

$$\lambda_N \;=\; \tfrac{1}{2}\big(v\sqrt{D} + u\big) \;=\; \varepsilon_D. \tag{3.14}$$

Proof.

Step 1 (the level-selection identity). The norm condition $N(\varepsilon_D) = -1$ reads

$$u^2 - v^2 D \;=\; -4, \qquad \text{i.e.} \qquad u^2 + 4 \;=\; v^2 D, \tag{3.15}$$

and the trace of the square is

$$\mathrm{Tr}(\varepsilon_D^2) \;=\; (\varepsilon + \varepsilon')^2 - 2\varepsilon\varepsilon' \;=\; u^2 + 2. \tag{3.16}$$

Hence $t + 2 = v^2 D$, which is (3.11).

Step 2 (the level and the rung). The decomposition (3.4) of Proposition 3.1 is therefore the decomposition of $v^2 D$ into squarefree kernel times square, so $p = \mathrm{kernel}(D)$ and $n = v\sqrt{D/p}$, an integer — that is, (3.12) and (3.13).

Step 3 (the eigenvalue on the floor). For (3.14),

$$\lambda_N \;=\; \tfrac{1}{2}\big(n\sqrt{p} + \sqrt{n^2 p - 4}\,\big), \tag{3.17}$$

and, by (3.11),

$$n^2 p - 4 \;=\; v^2 D - 4 \;=\; u^2, \tag{3.18}$$

so $\lambda_N = \frac{1}{2}(v\sqrt{D} + u) = \varepsilon_D$. ∎

Verification. All four identities were checked for all 28 fundamental discriminants D with $5 \le D \le 204$ and $N(\varepsilon_D) = -1$ — exhaustive over that range, the units found by a search bounded at $v \le 3 \cdot 10^5$ — with 0 failures [App. A]. The population and eleven of its rows (u, v, t, p, n), including $D = 73$ with $(u, v) = (2136, 250)$, were subsequently reproduced by two further instruments written independently of the first [App. A].

Remark 3.4 (the kernel does real work). The law is not a prime-level law:

$$D \;=\; 40 \;=\; 10 \cdot 2^2, \qquad \mathrm{kernel}(D) \;=\; 10, \tag{3.19}$$

so the level is composite and the group is $\Gamma_0(10)^+$ with the full Fricke involution w_{10}, not a partial Atkin–Lehner involution.

Corollary 3.5 (the two-floor identity). *Under the hypotheses of Theorem 3.3 the same closed geodesic is recorded twice: on $\Gamma_0(p)^+$ with eigenvalue the fundamental unit ε_D itself, and on $X(1)$ with eigenvalue ε_D^2. The floor realizes $\mathrm{Reg}\,\mathbb{Q}(\sqrt{D})$; $X(1)$ realizes $2\,\mathrm{Reg}\,\mathbb{Q}(\sqrt{D})$.*

The digit-exact instance of Corollary 3.5, at $D = 8$, is taken up in §6, where the doubling is read as a statement about lengths rather than about eigenvalues.

3.3 Which classes descend

The law of §3.2 names a trace. Which of the classes at that trace descends is settled by one algebraic identity in one direction and by a four-step construction in the other.

> **Theorem 3.6 (the descent criterion).** *Fix an integer $t \geq 3$ and write $t+2 = p\,n^2$ with p squarefree, as in (3.4). Let $\mathcal{Q}$ be a class of binary quadratic forms of discriminant t^2-4. Then*
>
> *(a) (necessity) if $\mathcal{Q}$ descends from $\Gamma_0(p)^+$, then n divides the content of $\mathcal{Q}$;*
>
> *(b) (sufficiency) conversely, if n divides the content of $\mathcal{Q}$, then $\mathcal{Q}$ descends from $\Gamma_0(p)^+$.*
>
> *Both statements are relative to the level p — the squarefree part of $t+2$, which by Theorem 3.3 is* $\mathrm{kernel}(D)$ *whenever t arises from a discriminant D with $N(\varepsilon_D) = -1$ — and not to admissible levels in general; see §3.4.*

Proof of (a). Write a Fricke-coset element as

$$N \;=\; \frac{1}{\sqrt{p}}\begin{pmatrix} pa & b \\ pc & pd \end{pmatrix}, \qquad pad - bc \;=\; 1, \tag{3.20}$$

so that $\operatorname{tr} N = \sqrt{p}\,(a+d)$ and $n = a+d$. Then

$$N^2 \;=\; \begin{pmatrix} pa^2 + bc & bn \\ pcn & bc + pd^2 \end{pmatrix}, \tag{3.21}$$

and the fixed-point form of N^2 is

$$\big(pcn,\;\; pn(d-a),\;\; -bn\big) \;=\; n \cdot \big(pc,\;\; p(d-a),\;\; -b\big). \tag{3.22}$$

The form of a descended class is thus n times a form, and its content is divisible by n. ■

Proof of (b), in four steps. Let $\mathcal{Q}$ have discriminant $\Delta = t^2 - 4$ with $n \mid \mathrm{cont}(\mathcal{Q})$, and put $q = \mathcal{Q}/n$, of discriminant

$$\frac{t^2-4}{n^2} \;=\; p\,(pn^2-4), \qquad \text{using} \qquad t^2 - 4 \;=\; (t-2)(t+2) \;=\; (pn^2-4)(pn^2). \tag{3.23}$$

Step 1 (q properly represents a multiple of p). From

$$4A\,q(x,y) \;=\; (2Ax+By)^2 - \Delta y^2 \tag{3.24}$$

and $p \mid \Delta$ we get

$$4A\,q(x,y) \;\equiv\; (2Ax+By)^2 \pmod{p}; \tag{3.25}$$

solving $2Ax + By \equiv 0 \pmod{p}$ is a single linear condition and has a coprime solution, and if $p \mid A$ already one takes $(x, y) = (1, 0)$.

Step 2 (transport). Complete (x, y) to a unimodular matrix. The transformed form (A'', B'', C'') has $p \mid A''$, and then $p \mid B''$ automatically:

$$B''^2 - 4A''C'' \;=\; \Delta \;\equiv\; 0 \pmod{p} \tag{3.26}$$

forces $p \mid B''^2$, and p is squarefree.

Step 3 (integrality is automatic). Write $A'' = pc$, $B'' = pB_1$, $C'' = -b$, and set

$$a \;=\; \frac{n - B_1}{2}, \qquad d \;=\; \frac{n + B_1}{2}. \tag{3.27}$$

The discriminant relation transported to the new form reads

$$p^2 B_1^2 \;+\; 4pcb \;=\; p\,(pn^2 - 4), \tag{3.28}$$

and dividing by p and rearranging,

$$p\,(n^2 - B_1^2) \;=\; 4 \;+\; 4cb \;\equiv\; 0 \pmod{4}. \tag{3.29}$$

Whether p is odd or $p \equiv 2 \pmod{4}$, (3.29) forces $n \equiv B_1 \pmod{2}$, so the two quantities in (3.27) are integers.

Step 4 (the unit condition is automatic). By (3.27) and (3.29),

$$p\,ad \;-\; bc \;=\; \frac{p\,(n^2 - B_1^2)}{4} \;-\; bc \;=\; (1 + cb) - bc \;=\; 1. \tag{3.30}$$

Hence

$$N \;=\; \frac{1}{\sqrt{p}} \begin{pmatrix} pa & b \\ pc & pd \end{pmatrix} \tag{3.31}$$

is a Fricke-coset element of trace $n\sqrt{p}$; N^2 has trace $n^2 p - 2 = t$, determinant 1 and lower-left entry $\equiv 0 \pmod{p}$; and its fixed-point form is

$$n \cdot \big(pc,\; p(d - a),\; -b\big) \;\sim\; n\,q \;=\; \mathcal{Q}. \tag{3.32}$$

■

No step of Steps 1–4 mentions t. The argument is uniform in the trace, and the machine ranges below are controls on it rather than its evidence.

Verification. Part (a) was tested over $3 \le t \le 30$ with the search complete in (b, c) over all divisor pairs for $|a| \le 30$ and the descent set saturated ($|a| \le 10$ and $|a| \le 30$ return identical counts): no descending class violates $n \mid$ content, and no class with $n \mid$ content fails to descend [App. A]. A later and independent control on (b) ran over $3 \le t \le 200$: 4,237 classes examined by exhaustive reduced-form cycles, 2,464 of them with $n \mid$ content, a descent explicitly constructed for every one of those, 0 failures, each N^2 verified exactly for $\det = 1$, $\mathrm{tr} = t$ and lower-left $\equiv 0 \pmod{p}$; the two symbolic identities of part (a) were re-verified on every admissible (a, b, c, d, p) in $[-3, 3]^4 \times \{2, 3, 5, 6, 7, 10\}$ [App. A].

Example 3.7 (trace 6). Here

$$t + 2 \;=\; 8 \;=\; 2 \cdot 2^2, \qquad \text{so} \qquad p = 2, \quad n = 2. \tag{3.33}$$

The three classes of discriminant 32 are:

reduced form	content	descends from $\Gamma_0(2)^+$
$(-4,\ 4,\ 1)$	1	no
$(-2,\ 4,\ 2)$	2	yes
$(-1,\ 4,\ 4)$	1	no

Exactly one content is even, so exactly one class descends from $\Gamma_0(2)^+$: the class of the symmetric form $(-2, 4, 2)$. The same selection had been obtained independently by a content-invariance argument using two 2×2 matrix multiplications and a conjugation inside $\Gamma_0(2)^+$; that route and the one here — reduction cycles of indefinite forms together with the identity of Theorem 3.6(a) — share no step.

Remark 3.8 (the shape of the descent). Over $3 \le t \le 30$: every class descends exactly at the traces with $n = 1$, namely

$$t \in \{3, 4, 5, 8, 9, 11, 12, 13, 15, 17, 19, 20, 21, 24, 27, 28, 29\}; \tag{3.34}$$

some class fails to descend exactly at the traces with $n \ge 2$, namely

$$t \in \{6, 7, 10, 14, 16, 18, 22, 23, 25, 26, 30\}; \tag{3.35}$$

and no trace in that range fails to have a descending class.

3.4 Level-relativity, and the multiplicity of admissible levels

A square root of a trace-t element has trace s with

$$s^2 - 2 \;=\; t, \qquad \text{hence} \qquad s^2 \;=\; t + 2; \tag{3.36}$$

and an Atkin–Lehner coset at an exact divisor $Q \parallel N$ has trace set $\sqrt{Q} \cdot \mathbb{Z}$ [7]. The levels at which a square root of a trace-t element can live therefore solve the *trace equation*

$$Q\,n^2 \;=\; t+2, \qquad Q,n\in\mathbb{Z}_{>0}. \tag{3.37}$$

Nothing forces Q to be squarefree, and nothing forces the solution to be unique.

Proposition 3.9 (multiplicity at trace 6). *Over the range $Q \le 500$, $n \le 59$ the trace equation (3.37) has the unique solution $(Q,n) = (5,1)$ at $t = 3$, and exactly two solutions at $t = 6$, namely $(Q,n) = (2,2)$ and $(Q,n) = (8,1)$. Both of the latter are realised by group elements: a search over levels $N \le 60$, exact divisors $Q \,\|\, N$ and matrix entries of absolute value at most 12 — exhaustive over that box — returns six witnesses, four at $Q = 2$ with fixed form of content 2 and two at $Q = 8$ with fixed form of content 1. Explicitly, at $Q = 8$:*

$$A \;=\; \begin{pmatrix} 8 & -1 \\ 8 & 0 \end{pmatrix}, \qquad \det A \;=\; 8, \tag{3.38}$$

$$A^2 \;=\; \begin{pmatrix} 56 & -8 \\ 64 & -8 \end{pmatrix}, \qquad \tfrac{1}{8}A^2 \;=\; \begin{pmatrix} 7 & -1 \\ 8 & -1 \end{pmatrix} \in \Gamma_0(8), \tag{3.39}$$

of determinant 1 and trace 6, with fixed-point form $(8,-8,1)$ of discriminant 32 and content **1**.

Verification. The instrument shares no code with the one used for Theorem 3.3: determinant, product, trace, the fixed-point form and the exact-divisor test are separately implemented. It was run first as a control at $t = 3$, where it reproduced the unique solution $(5,1)$ and the associated witness

$$A \;=\; \begin{pmatrix} 5 & -1 \\ 5 & 0 \end{pmatrix}, \qquad \tfrac{1}{5}A^2 \;=\; \begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix} \in \Gamma_0(5) \tag{3.40}$$

with fixed form $(5,-5,1)$ of content 1, before being run at $t = 6$; 10 checks, 10 pass, 0 fail [App. A]. The search box is a recall bound, not a sample: the enumeration is exhaustive inside it and silent outside it.

Since $n = 2$ at $t = 6$ and $2 \nmid 1$, the level-8 witness of Proposition 3.9 is a descent of a trace-6 class whose content is *not* divisible by n. This settles the quantifier in Theorem 3.6.

Remark 3.10 (the level-relative reading). The squarefree part p of $t+2$ — the kernel level kernel(D), in the setting of Theorem 3.3 — is the **minimal** admissible level of the trace equation: verified minimal at every trace at which admissible levels were found, over levels $N \le 60$, and **not the only** one. Accordingly, every unqualified "descends" in §§3.2–3.3 is to be read as "descends from $\Gamma_0(\mathrm{kernel}(D))^+$", and the necessity of Theorem 3.6(a) is false as an unquantified statement about descents from arbitrary admissible levels. What is untouched by this: the level-relative law itself; the minimality of the kernel level over the stated range; the descent identity (3.22) of Theorem 3.6(a), which is a two-line symbolic derivation; and the identity (3.11) over the 28-member population.

The multiplicity has a structural explanation, and it is not that two geodesics share a trace.

Proposition 3.11 (one geodesic, two levels). *At $t = 6$ the two admissible pairs give the same coset trace,*

$$1 \cdot \sqrt{8} \;=\; 2 \cdot \sqrt{2} \;=\; 2.828427125\dots, \tag{3.41}$$

exactly. Moreover $h(8) = 2$, where $h(N)$ denotes the largest divisor of 24 with $h^2 \mid N$, so that $\operatorname{Nor}\Gamma_0(8)$ is conjugate to $\Gamma_0(8/4)^+ = \Gamma_0(2)^+$ (Proposition 3.12(ii)). The two levels therefore sit inside one common normalizer, and the same closed geodesic is the first rung at level 8 and the second rung at level 2: the multiplicity of Proposition 3.9 is a multiplicity of (Q, n)-factorisations of one trace, not a multiplicity of objects.

Verification. 11 checks, 11 pass, 0 fail [App. A].

It remains to check that no *further* levels enter from the normalizers themselves.

Proposition 3.12 (the normalizer contributes no new level). *The normalizer of $\Gamma_0(N)$ in $\mathrm{PSL}(2,\mathbb{R})$ is $\Gamma_0(\nu|h)^+$, with h the largest divisor of 24 such that $h^2 \mid N$ and $\nu = N/h$, its elements being*

$$\frac{1}{\sqrt{e}}\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}, \qquad ade - bc\,N/(eh^2) = 1, \qquad e \parallel N/h^2 \tag{3.42}$$

[9, 10]. (The refinement is Conway's $n|h$; the letter is ν here because n denotes the rung index throughout this section.) Then

(i) the h-refinement is invisible to the trace: the trace is

$$\sqrt{e}\,(a + d) \;\in\; \sqrt{e}\cdot\mathbb{Z}, \tag{3.43}$$

the entry b/h never reaching the diagonal;

(ii) $\Gamma_0(\nu|h)$ is conjugate to $\Gamma_0(N/h^2)$ — conjugation by $\operatorname{diag}(1, h)$ sends

$$\begin{pmatrix} a & b/h \\ c\nu & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c\nu/h & d \end{pmatrix}, \qquad \nu/h \;=\; N/h^2; \tag{3.44}$$

hence $\operatorname{Nor}\Gamma_0(N)$ is conjugate to $\Gamma_0(N/h^2)^+$ with its full Atkin–Lehner group.

Consequently, over levels $N \le 60$ — exhaustively, the sweep not having been run beyond — the $\nu|h$ normalizer contributes no trace, no closed geodesic and no admissible level that the Atkin–Lehner sweep at level $M = N/h^2$ does not already carry.

Parts (i) and (ii) are derivations and carry no bound; what is bounded is the enumeration. Exactly 20 of the 60 levels $N \le 60$ have $h > 1$, namely

$$N \in \{4, 8, 9, 12, 16, 18, 20, 24, 27, 28, 32, 36, 40, 44, 45, 48, 52, 54, 56, 60\}, \tag{3.45}$$

and for each of them the normalizer drops to level $M = N/h^2$.

Verification. A brute-force control at $N \in \{4, 8, 9, 12, 16, 18, 25, 27, 36, 48, 49, 50\}$ over $|a|, |b|, |d| \le 6$ — a recall bound, not a sample — found that every normalizer trace carries a coset label e which is an exact divisor of $M = N/h^2$; the control sits on top of the conjugation derivation of (ii), not in place of it [App. A].

3.5 The arithmetic under the halving

The correspondence of §3.1 halves a length at every trace. What does not survive the halving uniformly is the arithmetic.

Proposition 3.13 (degree dichotomy). *With N as in Proposition 3.1 and λ_N its larger eigenvalue,*

$$\lambda_N + \lambda_N^{-1} = \sqrt{t+2}, \qquad \lambda_N - \lambda_N^{-1} = \sqrt{t-2}, \tag{3.46}$$

so that

$$\mathbb{Q}(\lambda_N) \supseteq \mathbb{Q}(\sqrt{t+2}, \sqrt{t-2}), \tag{3.47}$$

and hence λ_N is quadratic over $\mathbb{Q}$ if and only if one of $t \pm 2$ is a perfect square. In particular λ_N is quadratic in the norm-(-1) family, where $t-2 = u^2$ always and $\lambda_N = \varepsilon_D$, and in the degenerate branch $p = 1$ of Remark 3.2, where $t+2 = n^2$ and $\lambda_N \in \mathbb{Q}(\sqrt{t^2-4})$; in every other case λ_N has degree 4 — in particular at each of eight fundamental discriminants with $N(\varepsilon_D) = +1$ (those enumerated in the Verification below), where $\lambda_N = \sqrt{\varepsilon_D}$.

Verification. The criterion "$t \pm 2$ a perfect square" matches $\deg \lambda_N = 2$ with 0 mismatches over every integer trace $3 \le t \le 200$ [App. A]. The eight discriminants with $N(\varepsilon_D) = +1$ are $D \in \{12, 21, 24, 28, 33, 44, 56, 57\}$, and the eigenvalue is a quartic surd at each — for instance $D = 12$, $t = 4$, $\lambda_N = \sqrt{2 + \sqrt{3}}$.

Remark 3.14. The floor halves the length in both families; what splits is the arithmetic beneath the halving. On the norm-(-1) side the eigenvalue on the floor is ε_D, a unit of the *same* quadratic order, and the doubling downstairs is precisely the step $\varepsilon \mapsto \varepsilon^2$. On the norm-$(+1)$ side it is $\sqrt{\varepsilon_D}$, a unit of a quartic field: the halving is geometry with no number theory under it.

Remark 3.15 (the systole is not carried). Nor does the correspondence carry the systole. Over the 28 fundamental discriminants D with $5 \le D \le 204$ and $N(\varepsilon_D) = -1$, the element N of Theorem 3.3 realizes the systole of its floor for exactly $D \in \{5, 8, 13, 53\}$; the first failure is at $D = 17$. Two distinct mechanisms account for the failures. (1) $n \ge 2$ with $\sqrt{p} > 2$: the element N sits on the n-th rung while the first rung is already hyperbolic and shorter. At $D = 17$ the floor trace is $2\sqrt{17}$, and $\sqrt{17}$ is itself a closed geodesic of $\Gamma_0(17)^+$. This is exactly what does not happen at $D = 8$, where $n = 2$ but $\sqrt{2} < 2$ is elliptic — the order-4 cone point — so that the first rung is not a geodesic at all and the second wins. (2) An integer trace of $\Gamma_0(p)$ beats $\sqrt{p}$: at $D = 29$ the floor trace is $\sqrt{29} \approx 5.385$, but $\Gamma_0(29)$ contains an element of trace 3, since 5 is a quadratic residue mod 29. §6 carries the hypothesis this leaves standing, and §11 names what remains open.

4. The single-inequality dichotomy

A few of the floors constructed in §3 are Hecke triangle groups and all the rest are not, and the difference between the two families is not one of degree. The first rung of $\Gamma_0(p)^+$ has trace $\sqrt{p}$; the parabolic threshold is 2; and the comparison of those two numbers decides, on its own, the orders of the floor's elliptic points, its signature, whether it belongs to the Hecke family at all, which rung carries its shortest closed geodesic, and four further things scattered across this paper. This section isolates that comparison, states the eight distinct jobs it does, and shows that the list of levels it returns — $p \in \{1, 2, 3\}$ — is Takeuchi's list of arithmetic Hecke groups and Niven's theorem, in other clothing. The inequality is also silent about several things it is easily mistaken for deciding, and §4.5 says which.

4.1 The threshold

By §3.1 the Fricke coset of $\Gamma_0(p)^+$ has trace set $\sqrt{p} \cdot \mathbb{Z}$ [7], the k-th rung being the coset element of trace

$$t_k \;=\; k\sqrt{p}, \qquad k \in \mathbb{Z}. \tag{4.1}$$

An element of $\mathrm{PSL}(2, \mathbb{R})$ is elliptic, parabolic or hyperbolic according as $|t| < 2$, $|t| = 2$ or $|t| > 2$, so rung k is elliptic exactly when

$$k^2 p \;<\; 4. \tag{4.2}$$

Write $n_0(p)$ for the index of the least hyperbolic rung,

$$n_0(p) \;=\; \min\{\, k \geq 1 \;:\; k^2 p > 4 \,\}. \tag{4.3}$$

Proposition 4.1 (the threshold). *Let $p \geq 1$ be squarefree. The first rung of $\Gamma_0(p)^+$ is elliptic if and only if $\sqrt{p} < 2$, that is, if and only if $p \in \{1, 2, 3\}$; it is hyperbolic for every squarefree $p \geq 5$. Accordingly $n_0(p) = 2$ for $p \in \{2, 3\}$ and $n_0(p) = 1$ for $p \geq 5$. In the elliptic case the rung has finite order:*

$$\sqrt{2} \;=\; 2\cos(\pi/4), \qquad \sqrt{3} \;=\; 2\cos(\pi/6), \tag{4.4}$$

so the order is 4 at $p = 2$ and 6 at $p = 3$ (and 3 at the degenerate level $p = 1$, where $w_1 = \mathrm{id}$ by (3.7), the coset is the whole group, and $1 = 2\cos(\pi/3)$ is the trace of an elliptic element of $\mathrm{PSL}(2, \mathbb{Z})$; see §1). In the hyperbolic case the only elliptic trace in the Fricke coset is 0, of order 2.

Proof. Squarefreeness excludes $p = 4$, so $\sqrt{p} < 2$ is $p \leq 3$; (4.1) and (4.2) then give the trichotomy and the two values of n_0, since $p \leq 3 < 4 \leq 4p$ at $p \in \{2, 3\}$ and $p > 4$ at $p \geq 5$. The orders are read off (4.4). If $\sqrt{p} > 2$ then $|k\sqrt{p}| < 2$ forces $k = 0$. ■

The threshold enters the systole question through one further quantity. Let $t_{\min}(p)$ be the least integer trace exceeding 2 realised in $\Gamma_0(p)$.

Remark 4.2 (the two progressions). The trace set of $\Gamma_0(p)^+$ above the parabolic threshold is the union of two progressions, the integer traces of $\Gamma_0(p)$ and the hyperbolic rungs, and no tie between them is possible, $k\sqrt{p}$ being irrational for squarefree $p > 1$. Hence the systolic trace of the floor is

$$\min\big(t_{\min}(p),\ n_0(p)\sqrt{p}\big), \tag{4.5}$$

and rung n is the floor's shortest closed geodesic exactly when $n = n_0(p)$ and $n_0(p)\sqrt{p} < t_{\min}(p)$. The threshold fixes the second argument of (4.5); which argument wins is a separate question, taken up at $p = 2, 3$ in §5 and at general p in §11.

4.2 From the threshold to the arithmetic-Hecke list

The chain from (4.2) to the Hecke family is three steps long: the rung's order, the signature it forces, and the classification of genus-0 two-cone-point groups.

Proposition 4.3 (elliptic orders on the floor). *At $p = 2$ the group $\Gamma_0(2)$ realises no element of order 3 — an order-3 element has trace ± 1, and $\Gamma_0(2)$ has no element of odd trace at all (§5) — while the Fricke coset realises order 4 by Proposition 4.1; the elliptic orders of $\Gamma_0(2)^+$ are therefore contained in $\{2, 4\}$. At $p = 3$, $\Gamma_0(3)$ has no element of order 2, since $-4 \equiv 2 \pmod 3$ is not a square mod 3, and does have elements of order 3; the Fricke coset contributes order 6; the orders are $\{2, 3, 6\}$, the order-6 element absorbing an order-2 and an order-3 element at the same fixed point. At $p = 5$, $\Gamma_0(5)$ has no element of order 3, since $-3 \equiv 2 \pmod 5$ is not a square mod 5, and the Fricke coset's only elliptic trace is 0; every elliptic element of $\Gamma_0(5)^+$ has order exactly 2.*

Gauss–Bonnet for a Fuchsian group of genus g with c cusps and elliptic orders $e_1, \dots, e_r$ reads

$$\text{area} \;=\; 2\pi\Big(2g - 2 + c + \sum_i \big(1 - \tfrac{1}{e_i}\big)\Big). \tag{4.6}$$

For prime p the group $\Gamma_0(p)$ has two cusps, 0 and ∞, and w_p swaps them, so $\Gamma_0(p)^+$ has one cusp; the areas are $\pi/2$, $2\pi/3$ and π at $p = 2, 3, 5$.

Theorem 4.4 (the signatures). *With the orders of Proposition 4.3, equation (4.6) has a unique solution at each of the three levels, namely*

level	area	cusps	signature
$\Gamma_0(2)^+$	$\pi/2$	1	$(0;\ 2, 4;\ 1)$
$\Gamma_0(3)^+$	$2\pi/3$	1	$(0;\ 2, 6;\ 1)$
$\Gamma_0(5)^+$	π	1	$(0;\ 2, 2, 2;\ 1)$

Proof. With one cusp and the areas above, (4.6) reduces at the three levels to

$$2a + 3b \;=\; 5, \qquad \sum_i \big(1 - \tfrac{1}{e_i}\big) \;=\; \tfrac{4}{3}, \qquad 4g + k \;=\; 3, \tag{4.7}$$

where a and b count the points of order 2 and 4 at $p = 2$, and k the points of order 2 at $p = 5$; genus 0 is forced in each case, since $g \geq 1$ would make the elliptic contribution to (4.6) negative. The first relation has the unique non-negative solution $(a, b) = (1, 1)$ and the third $k = 3$. The second admits over the orders $\{2, 3, 6\}$ of Proposition 4.3 exactly two multisets, $\{2, 6\}$ and $\{3, 3\}$, of which the second is excluded by the order-6 element. ■

Two cone points at $p = 2$ and $p = 3$; three at $p = 5$. A genus-0 group with two cone points and one cusp is a triangle group, and its signature determines it up to conjugacy.

Theorem 4.5 (the floors run out at $p = 3$). *Let $p \geq 1$ be squarefree. If $\sqrt{p} < 2$, that is if $p \in \{1, 2, 3\}$, then $\Gamma_0(p)^+$ is a Hecke triangle group:*

$$\Gamma_0(1)^+ = G(3) = X(1), \qquad \Gamma_0(2)^+ = G(4), \qquad \Gamma_0(3)^+ = G(6). \tag{4.8}$$

The first floor with $\sqrt{p} > 2$ is not: $\Gamma_0(5)^+$ and $G(5)$ differ in area (π against $3\pi/5$), in torsion ($\{2, 2, 2\}$ against $\{2, 5\}$) and in abstract group type ($\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ against $\mathbb{Z}/2 * \mathbb{Z}/5$). The three groups in (4.8) are exactly the three finite entries of Takeuchi's list $q \in \{3, 4, 6, \infty\}$ of arithmetic Hecke groups [1], the entry $q = \infty$ being the degenerate one — so no level beyond $p = 3$ contributes, and $\Gamma_0(p)^+$ is a Hecke group precisely when its signature has two cone points, that is, precisely when $p < 4$.

Proof. At $p = 2$ and $p = 3$, Theorem 4.4 gives the signatures $(0; 2, 4; 1)$ and $(0; 2, 6; 1)$, which are those of $G(4)$ and $G(6)$; the identification follows from the classification of genus-0 two-cone-one-cusp groups. At $p = 1$, $w_1 = \mathrm{id}$ and $\Gamma_0(1)^+ = \mathrm{PSL}(2, \mathbb{Z})$ by (3.7). At $p = 5$ the signature has three cone points, so the group is not a triangle group at all. The closure of the list at $p = 3$ is Takeuchi's classification [1] read from the Fricke side. ■

Remark 4.6 (the near-miss at $p = 3$). The identification $\Gamma_0(3)^+ = G(6)$ is the one place in the chain where the naive computation gives a wrong answer. Reading the elliptic orders off the even coset alone yields $\{2, 3\}$ and, at area $2\pi/3$ with one cusp,

$$3a + 4b \;=\; 8, \qquad (a, b) \;=\; (0, 2), \qquad \text{signature } (0;\ 3, 3;\ 1), \tag{4.9}$$

which is not $G(6)$. The orders must be read from *both* cosets and then maximised at each fixed point; it is the Fricke coset that supplies the order-6 element, and it supplies it because $\sqrt{3} < 2$.

Remark 4.7 (what the theorem explains). Theorem 4.5 does not record a coincidence, it removes one. $G(4)$ and $G(6)$ are the arithmetic Hecke groups *because* they are the Fricke floors $\Gamma_0(2)^+$ and $\Gamma_0(3)^+$, and they are Fricke floors *because* $\sqrt{2}$ and $\sqrt{3}$ lie below the parabolic threshold. The golden case is where the supply stops: a floor whose systolic eigenvalue is $\varepsilon_5 = \varphi$ exists (§3), but it is $\Gamma_0(5)^+$, and $\sqrt{5} > 2$ puts it outside the Hecke family. The Hecke floors ran out at $p = 3$.

4.3 The eight jobs

Proposition 4.1 is used at eight distinct places in this paper. They are collected here because no one of them is visible from any other, and because the eight together are what the phrase "the single-inequality dichotomy" names.

#	what it decides	$\sqrt{p} < 2$	$\sqrt{p} > 2$	where
1	the floor's elliptic orders	rung 1 has order 3, 4 or 6	rung orders drop to 2	Prop. 4.3
2	the signature	two cone points	three ($p = 5$)	Thm. 4.4
3	the Hecke family	$X(1)$, $G(4)$, $G(6)$	outside it	Thm. 4.5
4	orders along the systolic axis	alternating $4, 2$ ($p = 2$)	uniformly 2 ($p = 5$)	§6
5	the shortest hyperbolic rung	$n_0 = 2$, trace $2\sqrt{p}$	$n_0 = 1$, trace $\sqrt{p}$	(4.3)
6	whether §3's floor element is systolic	rung 2 still wins ($D = 8$)	rung 1 is shorter ($D = 17$)	§3
7	provability of the stage's systole	coset-graded, hence proved	no grading, hence open	§5
8	one progression, or two, or none	rungs $1, 2$ of $\sqrt{2} \cdot \mathbb{Z}$	no progression	§2

Jobs 1, 2, 3 and 5 are proved above; the entries in the last two columns of rows 2 and 4 are the instances at $p = 5$ and $p = 2$ respectively, not general statements. Jobs 4, 6 and 7 are stated where their objects are constructed, and are recorded here only as uses of Proposition 4.1. Job 6 is the first of the two mechanisms by which the systole half of the descent law fails: when $n \geq 2$ and $\sqrt{p} > 2$ the element N of Proposition 3.1 sits on rung n while rung 1 is already hyperbolic and shorter — which is exactly what does not happen at $D = 8$, where $n = 2$ but $\sqrt{2} < 2$ is elliptic; over the 28 fundamental discriminants D with $5 \leq D \leq 204$ and $N(\varepsilon_D) = -1$ the first failure is $D = 17$ (§3, Remark 3.15). Job 7 is why the systoles of $G(4)$ and $G(6)$ are proved rather than bounded: both are Fricke floors, so their trace sets are graded by the coset and a systole is a comparison of two explicit sequences, whereas $G(5)$ is not a floor and that comparison has no analogue (§5). Job 8 needs one sentence here, because the intermediate case is the informative one.

Remark 4.8 (the three progressions). Where two occurrences of one surd are to be compared, the threshold decides whether they can lie on one progression. At $\sqrt{2}$ they do: the order-4 cone point of $\Gamma_0(2)^+$ is rung 1 and the floor's systole is rung 2 of the same progression $\sqrt{2} \cdot \mathbb{Z}$, the two separated only by the wall $|t| = 2$, and the square of the second has trace 6 in $\Gamma_0(2)$. At $\sqrt{3}$ the floor $\Gamma_0(3)^+$ does carry a progression $\sqrt{3} \cdot \mathbb{Z}$ whose rung 1 is the order-6 point, but the second occurrence is an integer trace of $X(1)$, in the even coset: two progressions, not one. At $\sqrt{5}$ there is no progression to be rungs of, since $\sqrt{5} > 2$ leaves $\Gamma_0(5)^+$ outside the Hecke family and $G(5)$ is not a floor. How the resulting pairs are classified is settled in §2, not here; what is settled here is that the three cases are three, and that the inequality is what separates them.

4.4 One list, four descriptions

The list $\{3, 4, 6\}$ that Theorem 4.5 returns as levels is the same list the Hecke family returns as orders, and the identification is a single theorem of Niven's. Write λ_q for the Hecke translation parameter,

$$\lambda_q \;=\; 2\cos(\pi/q), \tag{4.10}$$

so that

$$\lambda_q^2 \;=\; 4\cos^2(\pi/q) \;=\; 2+2\cos(2\pi/q). \tag{4.11}$$

Niven's theorem [11] states that the only rational values of $2\cos(r\pi)$ at rational r are

$$0,\ \pm 1,\ \pm 2. \tag{4.12}$$

Proposition 4.9. *For integers $q \geq 3$,*

$$\lambda_q^2 \in \mathbb{Q} \iff q \in \{3,4,6\}, \qquad \text{with} \qquad \lambda_3^2 = 1, \quad \lambda_4^2 = 2, \quad \lambda_6^2 = 3. \tag{4.13}$$

Proof. By (4.11) and (4.12), $\lambda_q^2 \in \mathbb{Q}$ if and only if $2\cos(2\pi/q) \in \{0, \pm 1, \pm 2\}$, and for $q \geq 3$:

q	3	4	5	6	≥ 7
$2\cos(2\pi/q)$	-1	0	$(\sqrt{5}-1)/2$	1	in $(1,2)$
rational	yes	yes	no	yes	no

The values in (4.13) follow from (4.11). ■

Verification. The list was re-instantiated rather than only cited: computing whether λ_q^2 is rational for every q with $3 \leq q \leq 12$ returns exactly $q \in \{3,4,6\}$, matching the published classification [App. A].

Corollary 4.10 (the two lists are one list). *In the notation of Theorem 4.5, the level and the Hecke order of a floor are related by*

$$p \;=\; \lambda_q^2, \tag{4.14}$$

the first rung of the floor being the elliptic generator of the Hecke group. Hence $q \in \{3,4,6\}$ corresponds to $p \in \{1,2,3\}$ term by term, and the two enumerations — the orders q at which λ_q^2 is rational, and the levels p at which the first rung is elliptic — are the same enumeration read through (4.14).

Proof. The floor $\Gamma_0(p)^+$ of Theorem 4.5 is $G(q)$ with $\sqrt{p} = \lambda_q$ by (4.4) and Proposition 4.1, whence (4.14); the values $1,2,3$ of (4.13) are the three levels. ■

Remark 4.11 (one theorem, four jobs). Niven's theorem appears in this paper at four places, and the four statements are not obviously the same statement. It forces the elliptic orders that $X(1)$ can carry, and through them the actualization laws of §8. It returns Takeuchi's arithmeticity list for the Hecke family [1]. It returns, by Corollary 4.10, the Fricke levels with $\sqrt{p} < 2$. And it returns the orders q at which the square of the minimal-trace element of $G(q)$ has integer trace, which is where the $q = 4$ construction of §5 stops at $q = 5$. Each is a different consequence of the rationality of $2\cos(2\pi/q)$.

4.5 What the threshold does not decide

The eight jobs of §4.3 are bounded on both sides, and two near neighbours of them are not jobs of the inequality at all.

Remark 4.12. The number of cusps is not decided by the threshold: $\Gamma_0(p)$ has two cusps for prime p and w_p swaps them, so every floor at prime level has exactly one, on both sides of the inequality. Neither is the *visibility* pattern along the systolic axis: at $p = 2$ the orders of the elliptic points alternate $4, 2$ and at $p = 5$ they are uniformly 2 — job 4 — but which of them $\mathrm{PSL}(2, \mathbb{Z})$ sees alternates identically at both levels, which is why the two discriminants are indistinguishable from $X(1)$ at that layer. §6 carries the distinction.

5. Systoles of the floors

A systole is a minimum taken over a whole group, and two of the three Hecke groups in the neighborhood give theirs up to arithmetic. Because $G(4)$ and $G(6)$ are the floors $\Gamma_0(2)^+$ and $\Gamma_0(3)^+$, their trace sets are graded by the Fricke coset, and the shortest closed geodesic is decided by setting one arithmetic progression against one congruence condition — a comparison quantified over the whole group rather than over a ball. The two comparisons come out the same way and for different reasons: at $p = 2$ the integer side is obstructed by a parity, at $p = 3$ by a quadratic non-residue, and neither obstruction is available at the other prime. The third Hecke group has no such grading, and its minimum is known anyway: on $G(5)$ the four syllable blocks of the free-product normal form are entrywise non-negative, the minimal hyperbolic trace is $2\varphi = 1 + \sqrt{5}$, and the interval $(2, 2\varphi)$ contains no trace at all. What the three results then make visible, in §5.5, is that the golden Hecke group and the golden floor are two different surfaces carrying two different systoles.

5.1 Traces, lengths, and the two graded groups

For a hyperbolic element of trace t the translation length of its axis is

$$\ell \;=\; 2\operatorname{arccosh}\big(|t|/2\big), \tag{5.1}$$

which is strictly increasing in $|t|$ [12]. We accordingly call the least $|t| > 2$ realized by a hyperbolic element of a Fuchsian group Γ its *systolic trace*, and the corresponding length its *systole*; by (5.1) the two determine each other, and every statement below is made at whichever of the two the record establishes. Write $\Theta(\Gamma)$ for the set of traces of elements of Γ, taken up to sign.

The restriction to hyperbolic elements is a convention and not a consequence of (5.1): the infimum of closed-curve lengths over all of Γ is 0 and unattained whenever Γ has a cusp, since the horocycles about it are arbitrarily short.

Let $q \geq 3$ and let $G(q)$ be the Hecke group generated by

$$S \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T \;=\; \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \qquad \lambda_q \;=\; 2\cos(\pi/q). \tag{5.2}$$

Two of the groups in the neighborhood are floors in the sense of §3: $G(4) = \Gamma_0(2)^+$ and $G(6) = \Gamma_0(3)^+$ (§4). For a floor the decomposition $\Gamma_0(p)^+ = \Gamma_0(p) \sqcup w_p\,\Gamma_0(p)$ grades the trace set by the coset [7],

$$\Theta\big(\Gamma_0(p)\big) \subseteq \mathbb{Z}, \qquad \Theta\big(w_p\,\Gamma_0(p)\big) \subseteq \sqrt{p}\cdot\mathbb{Z}, \tag{5.3}$$

so that

$$\Theta\big(G(4)\big) \;\subseteq\; \mathbb{Z} \cup \sqrt{2}\cdot\mathbb{Z}, \qquad \Theta\big(G(6)\big) \;\subseteq\; \mathbb{Z} \cup \sqrt{3}\cdot\mathbb{Z}. \tag{5.4}$$

A systole on either group is therefore a comparison of two explicit sequences.

Verification. Every trace of $G(4)$ was found to be an integer or an integer multiple of $\sqrt{2}$ over all cyclically reduced words of syllable length ≤ 8 — exhaustive at that depth, 335 distinct traces — and likewise for $G(6)$ with $\sqrt{3}$ at syllable length ≤ 7, 2,125 distinct traces [App. A].

> **Remark 5.1 (why the comparison is a proof).** Both halves of (5.3) quantify over the whole group: no ball, no enumeration bound, no syllable cap enters either statement. The coset grading is what converts a systole from a search into an arithmetic comparison, and it is available only because these two Hecke groups are also Fricke floors. §5.3 settles the third group by a different route, which needs no grading at all.

5.2 The systoles of $G(4)$ and $G(6)$

One side of the comparison is a congruence condition on the integer coset.

> **Lemma 5.2 (which integer traces the even coset realizes).** *Let p be prime and t an integer.*
>
> *(a) If p is odd, then $\Gamma_0(p)$ contains an element of trace t if and only if $t^2 - 4$ is a square modulo p.*
>
> *(b) If $p = 2$, then every element of $\Gamma_0(2)$ has even trace; in particular $\Gamma_0(2)$ has no element of odd trace at all.*

Proof of (a). Let $M = \left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right)$ lie in $\Gamma_0(p)$ with trace t. Then

$$a + d = t, \qquad ad \equiv 1 \pmod{p}, \tag{5.5}$$

since $ad - pbc = 1$; eliminating d, the entry a is a root of

$$a^2 - t\,a + 1 \equiv 0 \pmod{p}, \tag{5.6}$$

a quadratic whose discriminant is $t^2 - 4$. For p odd, (5.6) is solvable exactly when $t^2 - 4$ is a square modulo p. Conversely a solution a of (5.6), with $d = t - a$, lifts:

$$\begin{pmatrix} a & (ad-1)/p \\ p & d \end{pmatrix} \in \Gamma_0(p), \tag{5.7}$$

of determinant 1 and trace t. ■

Proof of (b). At $p = 2$ the passage from (5.5) to (5.6) is unavailable — completing the square requires 2 to be invertible, and modulo 2 the identity $(2a - t)^2 \equiv t^2 - 4$ reads $4ad \equiv 4$, which is vacuous. Argue instead from (5.5) directly:

$$ad \equiv 1 \pmod{2} \implies a, d \text{ both odd} \implies t = a + d \text{ even}. \tag{5.8}$$

■

Theorem 5.3 (the systoles of $G(4)$ and $G(6)$). *The systolic trace of $G(4)$ is $2\sqrt{2} = 2\lambda_4$ and the systolic trace of $G(6)$ is $2\sqrt{3} = 2\lambda_6$; in both cases the minimum is attained on the odd coset, at the class of R^2S, where $R = ST$. Equivalently the systole of $G(4) = \Gamma_0(2)^+$ is*

$$2\ln\bigl(1+\sqrt{2}\bigr) \;=\; 2\,\mathrm{Reg}\,\mathbb{Q}(\sqrt{2}). \tag{5.9}$$

Proof.

Step 1 (the integer coset at $p = 2$). By Lemma 5.2(b) every integer trace of $\Gamma_0(2)$ is even, so the least integer trace exceeding 2 is at most 4; and 4 is realized, by

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \qquad \det = 3-2 = 1, \qquad \text{lower-left} \equiv 0 \pmod{2}. \tag{5.10}$$

Step 2 (the integer coset at $p = 3$). Here the obstruction is Lemma 5.2(a). At $t = 3$,

$$t^2 - 4 \;=\; 5 \;\equiv\; 2 \pmod{3}, \tag{5.11}$$

and the squares modulo 3 are $\{0, 1\}$, so $\Gamma_0(3)$ has no element of trace 3. Its least integer trace exceeding 2 is therefore 4.

Step 3 (the odd coset). By (5.3) the odd-coset traces are the integer multiples of $\sqrt{p}$. The first rungs $\sqrt{2}$ and $\sqrt{3}$ are below the parabolic threshold 2 and hence elliptic (§4); the least hyperbolic odd-coset traces are therefore the second rungs,

$$2\sqrt{2} \;=\; 2.828427125\ldots, \qquad 2\sqrt{3} \;=\; 3.464101615\ldots \tag{5.12}$$

Step 4 (the comparison). Steps 1–3 give, on each group, two explicit candidates, and

$$\min\bigl(4,\ 2\sqrt{2}\bigr) \;=\; 2\sqrt{2}, \qquad \min\bigl(4,\ 2\sqrt{3}\bigr) \;=\; 2\sqrt{3}. \tag{5.13}$$

Both minima lie on the odd coset, at its second rung, which is the class of R^2S. ■

Remark 5.4 (the two proofs win for different reasons). The obstruction on the integer side is not the same obstruction at the two primes, and a single sentence covering both cases would hide that. At $p = 2$ it is a **parity**: $\Gamma_0(2)$ realizes no odd trace whatever, and the rung wins by

$$4 - 2\sqrt{2} \;=\; 1.1716\ldots \tag{5.14}$$

At $p = 3$ it is a **quadratic non-residue**: the trace 3 is excluded by (5.11) alone, and the rung wins by the smaller margin $4 - 2\sqrt{3}$. Two different obstructions, neither of them an absence of obstruction — which is why neither case generalises to the other for free.

Verification. At $p = 2$ a brute-force search over $|a|, |d| \le 40$ returns $\{4, 6, 8\}$ for the integer traces in $(2, 8]$ realisable in $\Gamma_0(2)$ [App. A]; independently, an exact trace-set computation on the floor returns $4, 6, 8, 10$ for the integer traces with $3 \le t < 12$, and 4 as the least integer trace above 2 [App. A]. The two agree, and neither uses the criterion of Lemma 5.2(a), which does not apply at $p = 2$.

5.3 The golden group: a minimal trace, and an interval that is empty

At $q = 5$ the generator parameter is the golden ratio itself,

$$\lambda_5 \;=\; 2\cos(\pi/5) \;=\; \varphi \;=\; \tfrac{1}{2}(1+\sqrt{5}), \tag{5.15}$$

and $G(5)$ is not a floor: by §4 the only Fricke floors that are Hecke groups are $\Gamma_0(2)^+ \;=\; G(4)$ and $\Gamma_0(3)^+ \;=\; G(6)$, and $\Gamma_0(5)^+ \;\neq\; G(5)$ by Proposition 5.8 below. There is accordingly no coset decomposition grading $\Theta(G(5))$, and the comparison of §5.2 has no analogue. What replaces it is the normal form. As a free product

$$G(q) \;=\; \langle S\rangle * \langle R\rangle \;\cong\; \mathbb{Z}/2 * \mathbb{Z}/q, \qquad R = ST \text{ of order } q, \tag{5.16}$$

[2], every hyperbolic conjugacy class has a cyclically reduced representative

$$W \;=\; R^{a_1}S\,R^{a_2}S\cdots R^{a_k}S, \qquad a_i \in \{1,\dots,q-1\}, \tag{5.17}$$

unique up to cyclic rotation [13, 14]; we write the class as the cyclic word $(a_1,\dots,a_k)$ and call k its syllable count. Since the trace is a conjugacy invariant, a bound on k is an exhaustive statement about conjugacy classes rather than a sample of the group — the distinction that separates the populations below from a ball of given radius.

> **Theorem 5.5 (the minimal hyperbolic trace of $G(5)$).** *The minimal hyperbolic trace of $G(5)$ is*
> $$|t|_{\min} \;=\; 2\varphi \;=\; 1+\sqrt{5} \;=\; 3.236067977500\dots, \tag{5.18}$$
> *attained exactly by the cyclic words* (2) *and* (3).

Proof. To the cyclic word $(a_1,\dots,a_k)$ attach the product $P = M_{a_1}\cdots M_{a_k}$ of the four syllable blocks, each taken with the sign that makes it entrywise non-negative, so that $|\operatorname{tr} W| = \operatorname{tr} P$. Expanding the trace as a sum over closed index paths,

$$\operatorname{tr} P \;=\; \sum_{i_0\to i_1\to\cdots\to i_k=i_0}\ \prod_{j=1}^{k}(M_{a_j})_{i_{j-1}i_j}, \tag{5.19}$$

and every term of (5.19) is non-negative, because every entry is. Put

$$m \;=\; \#\{\, i : a_i \in \{2,3\}\}. \tag{5.20}$$

The diagonal entries of the blocks are φ for $a \in \{2,3\}$ and 1 for $a \in \{1,4\}$, so the two constant paths of (5.19) contribute φ^m each and

$$|\operatorname{tr} W| \;=\; \operatorname{tr} P \;\geq\; 2\varphi^m. \tag{5.21}$$

Step 1 ($m \geq 1$). By (5.21),

$$|\operatorname{tr} W| \;\geq\; 2\varphi \;=\; 3.236067977500\ldots \;>\; 2, \tag{5.22}$$

so W is hyperbolic; and equality in (5.22) forces $m = 1$ with no further path contributing, that is $k = 1$ and the cyclic word (2) or (3).

Step 2 ($m = 0$). Then $a_i \in \{1, 4\}$ for every i. If the cyclic word is constant, W is the parabolic $T^{\pm k}$, of trace ± 2. Otherwise some $a_i = 1$ is adjacent to some $a_j = 4$; besides the two constant paths, each contributing 1, the path that steps $0 \to 1$ across that block M_1 (entry φ) and $1 \to 0$ across that block M_4 (entry φ), all remaining factors being diagonal entries 1, contributes φ^2. Hence

$$|\operatorname{tr} W| \;\geq\; 2 + \varphi^2 \;=\; 3 + \varphi \;=\; 4.618033988750\ldots \;>\; 2\varphi. \tag{5.23}$$

Comparing (5.22) and (5.23) gives (5.18) together with the two minimizing classes. ■

Verification. Exhaustively over all 4^k cyclically reduced words with syllable count $k = 1, \ldots, 11$ — 5,592,404 words visited — the minimal hyperbolic |trace| at syllable count k is exactly $(k+1)\varphi$ at $k = 1$ and at $3 \leq k \leq 11$, and $3+\varphi$ at $k = 2$, strictly increasing in k over that range; the minimizers are (2) at $k = 1$, $(1, 4)$ at $k = 2$, and $(1^{k-1}, 2)$ for $3 \leq k \leq 11$ [App. A]. **The minimality is proved above and not searched, and it rests on a single channel**: a second, independently written computation reached the minimizing class from outside and reproduced the element

$$W_0 \;=\; TST \;=\; \begin{pmatrix} \varphi & \varphi \\ 1 & \varphi \end{pmatrix}, \qquad \operatorname{tr} W_0 \;=\; 2\varphi, \tag{5.24}$$

together with its translation length 2.1225501238100713040660 38, its eigenvalue $\varepsilon = \varphi + \sqrt{\varphi}$ — a Salem number of degree 4 — its endpoint field $L = \mathbb{Q}(\sqrt{\varphi})$ with two real embeddings and one complex pair, and its count of 0 cone points per period; but that channel does not address minimality, and no part of Theorem 5.5 or Corollary 5.6 rests on it [App. A]. By Theorem 5.5 the element (5.24), having trace 2φ, represents one of the two minimizing classes.

> **Corollary 5.6 (the empty interval).** *No element of $G(5)$ has |trace| strictly between 2 and 2φ. Consequently*
>
> $$\sqrt{5} \;=\; 2\varphi - 1 \;=\; 2.236067977500\ldots \tag{5.25}$$
>
> *is an element of $\mathbb{Z}[\varphi]$ that is not a realized trace of $G(5)$ at all, and neither is 3, the minimal hyperbolic trace of the modular group, since $3 < 2\varphi$.*

Proof. Hyperbolic elements satisfy $|t| \geq 2\varphi$ by Theorem 5.5, elliptic elements $|t| < 2$ and parabolic elements $|t| = 2$; the three cases are exhaustive. Both $\sqrt{5}$ and 3 lie in the interval so excluded. ■

The gap is thus provable, not merely unobserved.

Verification. Over 11,661 distinct realized trace values sampled to syllable count $k \leq 9$, neither $\sqrt{5}$ nor 3 occurs [App. A]. The value $2+\varphi = 3.618033988750\ldots$ also failed to occur in that sample, but it lies **above** 2φ and is therefore a bounded negative only, not covered by Corollary 5.6. Independently and earlier, a ball of 34,942 elements of $G(5)$ — syllable length ≤ 14 in $\{S, T, T^{-1}\}$, 408 distinct traces — returned no element of trace $\sqrt{5}$, and recorded that this was not a proof [App. A].

Remark 5.7 (the integer traces). Corollary 5.6 excludes 3 but says nothing about integer traces above 2φ. The realized integer traces of $G(5)$ are $\{0, \pm 2\}$ over all cyclically reduced words of syllable length ≤ 12 — exhaustive over conjugacy classes at that depth, 22,369,620 words visited, 182,474 distinct traces — and no more than that is claimed here; §11 states what stands in the way of more.

5.4 The three groups together

For $q = 4$ and $q = 6$ the minimal hyperbolic trace is $2\lambda_q$ by Theorem 5.3, whose proof is the comparison (5.13) of two arithmetic sequences and is available only because those two Hecke groups are Fricke floors. For $q = 5$ the minimal hyperbolic trace is $2\lambda_5 = 2\varphi$ by Theorem 5.5, whose proof is the block non-negativity (5.19)–(5.21) and uses no coset grading, no congruence criterion and no floor.

Together they close the systole question for the family: each of the three Hecke groups $G(4)$, $G(5)$, $G(6)$ has minimal hyperbolic trace $2\lambda_q$, proved — by two mechanisms on the Fricke floors, and by a third, floor-free mechanism at $q = 5$.

The two results stand on their own evidence: Theorem 5.3 on Lemma 5.2 and the coset grading (5.3), Theorem 5.5 on the normal form (5.17) and the path expansion (5.19).

5.5 $G(5)$ and $\Gamma_0(5)^+$ are different surfaces

At $q = 4$ the Hecke group and the floor coincide. That is an accident of the signature, and it does not repeat.

Proposition 5.8 (the signatures). *$\Gamma_0(5)^+$ has signature $(0; 2, 2, 2; 1)$ and $G(5)$ has signature $(0; 2, 5; 1)$: they differ in area, in torsion, and in abstract group type,*

$$\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2 \quad \text{against} \quad \mathbb{Z}/2 * \mathbb{Z}/5, \tag{5.26}$$

and are therefore not the same group. At $p = 2$, by contrast, $\Gamma_0(2)^+$ has signature $(0; 2, 4; 1)$, which is the signature of $G(4)$.

Proof.

Step 1 (the cusp). For p prime, $\Gamma_0(p)$ has two cusps, 0 and ∞, and w_p interchanges them; hence $\Gamma_0(p)^+$ has one cusp.

Step 2 (the elliptic orders). By (5.3), an elliptic element has either an integer trace in $\{0, \pm 1\}$ on the even coset, or a trace $\sqrt{p}\,k$ with $|\sqrt{p}\,k| < 2$ on the odd coset. At $p = 5$ an element of order 3 would need -3 to be a square modulo 5; the squares are $\{0, 1, 4\}$ and $-3 \equiv 2$, so there is none, while the odd coset admits only $k = 0$, of trace 0 and order 2. Every elliptic element of $\Gamma_0(5)^+$ therefore has order exactly 2. At $p = 2$ order 3 is again unrealized, while the odd-coset trace $\pm\sqrt{2}$ has absolute value below 2 and gives order 4; the orders are $\{2, 4\}$.

Step 3 (Gauss–Bonnet). With

$$\text{area} \;=\; 2\pi\Big(2g - 2 + c + \textstyle\sum_i (1 - 1/e_i)\Big), \tag{5.27}$$

one cusp, and the orders of Step 2, the area π of $\Gamma_0(5)^+$ gives

$$4g + 2 + k \;=\; 5 \implies g = 0,\ k = 3 \tag{5.28}$$

uniquely, since $g \geq 1$ forces $k < 0$; and the area $\pi/2$ of $\Gamma_0(2)^+$ gives, with a and b the numbers of cone points of order 2 and 4,

$$2a + 3b \;=\; 5 \implies (a, b) = (1, 1) \tag{5.29}$$

uniquely. ■

A genus-zero group with two cone points and one cusp is a triangle group, and its signature determines it up to conjugacy; (5.29) therefore *derives* the identification $\Gamma_0(2)^+ \cong G(4)$ rather than importing it. The four objects side by side:

group	area	cusps	signature
$\Gamma_0(2)^+$	$\pi/2$	1	$(0; 2, 4; 1)$
$G(4)$	$\pi/2$	1	$(0; 2, 4; 1)$
$\Gamma_0(5)^+$	π	1	$(0; 2, 2, 2; 1)$
$G(5)$	$3\pi/5$	1	$(0; 2, 5; 1)$

The consequence for lengths is immediate and is the point of the section.

Corollary 5.9 (two golden surfaces, two systoles). *The systole of $G(5)$ is*

$$2 \operatorname{arccosh} \varphi \;=\; 2.122550124\ldots, \tag{5.30}$$

while the systolic trace of the floor $\Gamma_0(5)^+$ is $\sqrt{5}$ and its systole is

$$2 \ln \varphi \;=\; 2 \operatorname{Reg} \mathbb{Q}(\sqrt{5}) \;=\; 0.962423650\ldots \tag{5.31}$$

The golden Hecke group does not carry the golden regulator.

Nor does it carry any multiple of it.

Proposition 5.10 (no rung of the golden ladder). *A closed geodesic of length $2k \ln \varphi$ has eigenvalue φ^k and hence trace*

$$t_k \;=\; \varphi^k + \varphi^{-k}, \tag{5.32}$$

which is $\sqrt{5}\, F_k$ for k odd and the Lucas number L_k for k even:

$$t_1, \ldots, t_8 \;=\; \sqrt{5},\ 3,\ 2\sqrt{5},\ 7,\ 5\sqrt{5},\ 18,\ 13\sqrt{5},\ 47. \tag{5.33}$$

None of these is a realized trace of $G(5)$.

Proof and scope. At $k = 1$ and $k = 2$ the values $t_1 = \sqrt{5}$ and $t_2 = 3$ are excluded unconditionally by Corollary 5.6. The remaining non-realizations in (5.33) are ball-bounded: the remaining odd-k values are independent enumeration facts, and the even-k values follow from the integer-trace census of §5.3, which is exhaustive over conjugacy classes to syllable length 12 and not beyond [App. A]. ■

Remark 5.11 (where the $q = 4$ construction stops). The failure at $q = 5$ is one step earlier than any statement about commensurability. The element TST of trace $2\lambda_q$ has a square of trace

$$\operatorname{tr}\big((TST)^2\big) \;=\; 4\lambda_q^2 - 2, \tag{5.34}$$

which is 6 at $q = 4$ and 10 at $q = 6$, both integers, but

$$4\varphi + 2 \;\notin\; \mathbb{Z} \tag{5.35}$$

at $q = 5$. An integer trace is what would let the square be a class of $\mathrm{PSL}(2, \mathbb{Z})$ at all, so the construction is blocked before the rescaling that exhibits the class inside $\Gamma_0(p)^+$ is reached; and the block is not repairable by any choice of representative, since the trace is a conjugation invariant.

5.6 The systolic axis of $G(5)$

The last result of the section is about where the shortest geodesic sits rather than how long it is.

Proposition 5.12 (the cone points miss the axis, and $q = 4$ is the only exception). *For any Hecke group,*

$$TST \;=\; \begin{pmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{pmatrix}, \qquad \operatorname{tr}(TST) = 2\lambda, \tag{5.36}$$

and its axis is the semicircle of radius $\sqrt{\lambda^2 - 1}$ centred at 0. The two home cone points — i, fixed by S, of order 2, and $z_0 = e^{i\pi/q}$, fixed by ST, of order q — both lie on $|z| = 1$. Hence the home cone points lie on that axis if and only if

$$\lambda^2 - 1 \;=\; 1 \iff \lambda = \sqrt{2} \iff q = 4, \tag{5.37}$$

uniquely among all Hecke groups. At $q = 5$ the axis is the systolic axis, by Theorem 5.5, and it is cone-free.

q	λ_q	axis radius $\sqrt{\lambda_q^2 - 1}$	home cone point on the axis
3	1	0 (degenerate: TST is parabolic)	—
4	$\sqrt{2}$	1	yes
5	φ	$\sqrt{\varphi} = 1.272019650\ldots$	no
6	$\sqrt{3}$	$\sqrt{2}$	no
7	$2\cos(\pi/7)$	$\sqrt{1 + 2\cos(2\pi/7)}$	no

q	λ_q	axis radius $\sqrt{\lambda_q^2 - 1}$	home cone point on the axis

Verification. Over every elliptic fixed point in a ball at $q = 5$ — 595 points examined — the minimum distance to the systolic axis is $0.2406059125298017\ldots$, strictly positive [App. A]. The statement for the two home cone points is exact; the statement for their orbits is bounded by that ball.

That minimum is not a decimal.

Proposition 5.13 (the closest approach, exactly). *At $q = 5$,*

$$d\big(i,\ \text{the systolic axis of } G(5)\big) \;=\; \tfrac{1}{2}\ln\varphi \;=\; \tfrac{1}{2}\,\mathrm{Reg}\,\mathbb{Q}(\sqrt{5}). \tag{5.38}$$

Proof. For a point $z = x + iy$ and the semicircle $|w| = R$ centred at 0,

$$\cosh d\big(z, |w| = R\big) \;=\; \frac{|z|^2 + R^2}{2Ry}. \tag{5.39}$$

At $z = i$ and $R = \sqrt{\varphi}$, using $1 + \varphi = \varphi^2$,

$$\cosh d \;=\; \frac{1+\varphi}{2\sqrt{\varphi}} \;=\; \frac{\varphi^2}{2\sqrt{\varphi}} \;=\; \frac{\varphi^{3/2}}{2}, \tag{5.40}$$

while

$$\cosh\big(\tfrac{1}{2}\ln\varphi\big) \;=\; \frac{\varphi^{1/2} + \varphi^{-1/2}}{2} \;=\; \varphi^{-1/2}\,\frac{\varphi+1}{2} \;=\; \varphi^{-1/2}\,\frac{\varphi^2}{2} \;=\; \frac{\varphi^{3/2}}{2}. \tag{5.41}$$

The two agree, and cosh is injective on $[0, \infty)$. ■

Verification. Independently, from the other side of the neighborhood: the systolic axis misses the order-2 cone orbit by exactly

$$\tfrac{1}{2}\ln\varphi \;=\; 0.24060591252980172\ldots, \tag{5.42}$$

the minimum being attained at the top of the axis, $z = i\sqrt{\varphi}$; a geometric scan of one period at 40,000 sample points matches (5.42) to residual 4.70×10^{-10} [App. A]. The two computations share no step.

Remark 5.14 (what Proposition 5.13 does not say). A golden-ratio length *does* occur on $G(5)$: (5.38) is one. It is a distance from a cone point to a geodesic, not a member of the length spectrum, and Corollary 5.9 and Proposition 5.10 are statements about closed-geodesic lengths only. Read without that scope they would overclaim.

6. The doubling law and the bead ladder

Corollary 3.5 records one closed geodesic on two surfaces and reads the two records in eigenvalues. At $D = 8$ both records can be written down and compared digit by digit, and the comparison is a statement about lengths: the length on $X(1)$ is exactly twice the length on the floor, and the factor 2 is not an accident of the example but the visible face of an arithmetic obstruction — the order attached to the trace-6 class carries no unit of norm -1, so the geodesic has no choice but to run to ε^2. The axis underneath this identity carries more structure than its length. It is a necklace: the cone points on it sit at the integer multiples of the regulator, indexed by the powers of the fundamental unit, and the parity of the index decides which of them the modular group can see. That indexing is a theorem and not a tabulation — the stabiliser of the axis is infinite dihedral and its reflection set is exactly $\{\varepsilon^k\}$ — but it is a theorem with two hypotheses, and §6.4 says which floors have them and exhibits the test that decides an arbitrary one.

6.1 The doubling law at $D = 8$

Take $D = 8$ in Theorem 3.3. Then $\varepsilon_D = 1+\sqrt{2}$ has $(u, v) = (2, 1)$ and norm -1, the trace it determines on $X(1)$ is $t = u^2 + 2 = 6$, and the level-selection identity (3.11) reads

$$t + 2 \;=\; u^2 + 4 \;=\; v^2 D \;=\; 8 \;=\; 2 \cdot 2^2, \qquad \text{so} \qquad p = 2, \quad n = 2 \tag{6.1}$$

by (3.12) and (3.13). The two eigenvalues of Corollary 3.5 are

$$\varepsilon_8 \;=\; 1 + \sqrt{2} \;=\; 2.414213562\ldots, \qquad N(\varepsilon_8) = -1, \tag{6.2}$$

$$\lambda \;=\; \varepsilon_8^2 \;=\; 3 + 2\sqrt{2} \;=\; 5.828427125\ldots \tag{6.3}$$

Proposition 6.1 (the doubling law at $D = 8$). *The trace-6 class of* $\mathrm{PSL}(2, \mathbb{Z})$ *has translation length*

$$\ell_{X(1)} \;=\; 2\operatorname{arccosh}(6/2) \;=\; 4\ln(1 + \sqrt{2}) \;=\; 3.525494348\ldots, \tag{6.4}$$

while the Fricke-coset element N of trace $2\sqrt{2}$ *on* $\Gamma_0(2)^+$ *has*

$$\ell(N) \;=\; 2\ln(1 + \sqrt{2}) \;=\; 1.762747174\ldots \tag{6.5}$$

In regulators,

$$\ell(N) \;=\; 2\operatorname{Reg}\mathbb{Q}(\sqrt{2}), \qquad \ell_{X(1)} \;=\; 4\operatorname{Reg}\mathbb{Q}(\sqrt{2}) \;=\; 2\,\ell(N). \tag{6.6}$$

Proof. Both lengths are $\ell = 2\log\lambda$ at the respective eigenvalues (6.2) and (6.3), and $\lambda_{X(1)} = \lambda_{\text{floor}}^2$ by Corollary 3.5, which is the factor 2 in (6.6). For the first equality in (6.4),

$$\varepsilon_8^2 + \varepsilon_8^{-2} \;=\; (3 + 2\sqrt{2}) + (3 - 2\sqrt{2}) \;=\; 6, \tag{6.7}$$

so $\cosh\big(2\ln(1+\sqrt{2})\big) = 3$ and $2\operatorname{arccosh} 3 = 4\ln(1+\sqrt{2})$. ■

Verification. The identity $2\operatorname{arccosh} 3 = 4\ln(1+\sqrt{2})$ was checked numerically with residual 4.4×10^{-16} [App. A]. Independently, on the trace line rather than on the floor: the regulators at the four traces $t = 3, 4, 5, 6$ are $\ln\varphi = 0.481211825$, $\ln(2+\sqrt{3}) = 1.316957897$, $\ln\big((5+\sqrt{21})/2\big) = 1.566799237$ and $\ln(1+\sqrt{2}) = 0.881373587$, and over exactly those four traces $\ell = 4 \cdot (\text{regulator})$ at $t = 3$ and $t = 6$ while $\ell = 2 \cdot (\text{regulator})$ at $t = 4$ and $t = 5$ [App. A] — the two traces carrying the factor 4 being the two norm-(-1) traces of that range, in the sense of Proposition 3.13.

The factor is forced, and the mechanism is the conductor.

> **Proposition 6.2 (why the geodesic runs to ε^2).** *At $t = 6$,*
>
> $$t^2 - 4 \;=\; 32 \;=\; 2^2 \cdot 8, \tag{6.8}$$
>
> *and $f = 2$, $D_0 = 8$ is the only factorisation $32 = f^2 D_0$ with D_0 a fundamental discriminant. The order of discriminant 32 is*
>
> $$\mathcal{O}_{32} \;=\; \mathbb{Z}[2\sqrt{2}] \;=\; \mathbb{Z} + 2\,\mathbb{Z}[\sqrt{2}], \tag{6.9}$$
>
> *and*
>
> $$\varepsilon_8 = 1+\sqrt{2} \;\notin\; \mathcal{O}_{32}, \qquad \varepsilon_8^2 = 3+2\sqrt{2} \;\in\; \mathcal{O}_{32}. \tag{6.10}$$
>
> *Moreover $\mathcal{O}_{32}$ has no unit of norm -1. Hence its fundamental unit is ε_8^2, of norm $+1$, and that unit is the eigenvalue (6.3).*

Proof. The factorisation statement and (6.9)–(6.10) are direct: an element of $\mathcal{O}_{32}$ is $x + y\sqrt{2}$ with y even, which excludes ε_8 and admits ε_8^2. A unit of norm -1 in $\mathcal{O}_{32}$ would solve

$$x^2 - 8y^2 \;=\; -1, \tag{6.11}$$

whence

$$x^2 \;\equiv\; 7 \pmod{8}, \tag{6.12}$$

which is impossible, the squares mod 8 being $\{0, 1, 4\}$. ■

Verification. The factorisation was checked exhaustively over f; (6.11) was additionally swept over $y \leq 10^5$ with no solution; the powers (x, y) of ε_8 used in (6.10) are $(1,1), (3,2), (7,5), (17,12)$ in exact integer arithmetic [App. A].

> **Remark 6.3 (the length reading of the two-floor identity).** Corollary 3.5 states the two-floor identity in eigenvalues: the floor realizes $\operatorname{Reg}\mathbb{Q}(\sqrt{D})$ and $X(1)$ realizes $2\operatorname{Reg}\mathbb{Q}(\sqrt{D})$. Read in lengths, by $\ell = 2\log\lambda$, the same identity says the floor realizes $2\operatorname{Reg}\mathbb{Q}(\sqrt{D})$ and $X(1)$ realizes $4\operatorname{Reg}\mathbb{Q}(\sqrt{D})$; the doubling of the length *is* the squaring of the eigenvalue. Proposition 6.2 supplies what the eigenvalue statement leaves implicit: at $D = 8$ the square is not a choice. The companion instance one trace down is $t = 3$, where $\ell_{X(1)} = 4\ln\varphi = 4\operatorname{Reg}\mathbb{Q}(\sqrt{5})$ [App. A].

6.2 The axis as a necklace

Fix a floor $\Gamma_0(p)^+$ and let A be the axis of a Fricke-coset element N of trace $n\sqrt{p}$, as constructed in Proposition 3.1. Two words are used for the rest of the section, and both are this paper's abbreviations rather than standard terminology. A *bead* is an elliptic fixed point of $\Gamma_0(p)^+$ lying on A — a cone point of the quotient orbifold through which the closed geodesic passes — and the *necklace* is the set of beads with their orders. A bead is called *visible* when its stabiliser in $\Gamma_0(p)^+$ meets $\Gamma_0(p) \subset \mathrm{PSL}(2,\mathbb{Z})$ nontrivially, and *invisible* when its whole stabiliser lies in the Fricke coset; an invisible bead is a cone point of the floor that the modular group does not see at all. Finally, a *pivot* is a point of A fixed by an involution of $\mathrm{PSL}(2,\mathbb{Z})$.

Position on A is measured by arclength. For the geodesic that is the semicircle of centre e and radius r, writing $z = e + re^{i\theta}$ and $z = x + iy$,

$$ds = \frac{d\theta}{\sin\theta}, \qquad s = \ln\tan(\theta/2), \tag{6.13}$$

$$u(z) := \tan(\theta/2) = \frac{1-\cos\theta}{\sin\theta} = \frac{r-(x-e)}{y} = e^{s}. \tag{6.14}$$

The coordinate u is an algebraic number at every bead, so bead positions and spacings are logarithms of exact algebraic numbers and no numerical arccosh or arctan enters a load-bearing step. The normalisation below puts $u = 1$ at the fixed point of w_p itself. One further point of method: the computations of this section run in $\Gamma_0(p)^+$ coordinates on both floors and use no Hecke coordinates, so the $p = 2$ results do not assume the isomorphism $\Gamma_0(2)^+ \cong G(4)$ of §5 — the necklace on the Hecke side is reproduced here rather than cited.

Observation 6.4 (the two necklaces). *On the systolic axis of* $\Gamma_0(2)^+$ *every bead found sits at an exact integer multiple of* $\mathrm{Reg}\,\mathbb{Q}(\sqrt{2}) = \ln(1+\sqrt{2})$ *with* $u = \varepsilon_8^k$*, the orders alternating* 4 *and* 2*:*

k	$s = k\,\mathrm{Reg}$	order	$u = \varepsilon_8^k$	visible in $\mathrm{PSL}(2,\mathbb{Z})$
-3	-2.644121	4	$-7+5\sqrt{2}$	yes
-2	-1.762747	2	$3-2\sqrt{2}$	no
-1	-0.881374	4	$-1+\sqrt{2}$	yes
0	0.000000	2	1	no ($= w_2$)
1	0.881374	4	$1+\sqrt{2}$	yes
2	1.762747	2	$3+2\sqrt{2}$	no
3	2.644121	4	$7+5\sqrt{2}$	yes

Eleven consecutive k were recovered, $-5 \le k \le 5$*, with no exceptions, from a search box of* ± 60 *in the enumerating parameters. On the systolic axis of* $\Gamma_0(5)^+$ *the beads sit at exact integer multiples of* $\mathrm{Reg}\,\mathbb{Q}(\sqrt{5}) = \ln\varphi$ *with* $u = \varphi^k$*, and every order is* 2*:*

k	$s = k\,\mathrm{Reg}$	order	$u = \varphi^k$	visible in $\mathrm{PSL}(2,\mathbb{Z})$
-2	-0.962424	2	$(3-\sqrt{5})/2$	no
-1	-0.481212	2	$(-1+\sqrt{5})/2$	yes
0	0.000000	2	1	no ($= w_5$)
1	0.481212	2	$(1+\sqrt{5})/2 = \varphi$	yes
2	0.962424	2	$(3+\sqrt{5})/2 = \varphi^2$	no
3	1.443635	2	$2+\sqrt{5} = \varphi^3$	yes
4	1.924847	2	$(7+3\sqrt{5})/2 = \varphi^4$	no

Nineteen consecutive k were recovered, $-8 \le k \le 10$, from the same box. Per period of N^2 — that is, per $4\,\mathrm{Reg}$, the length (6.6) of the geodesic on $X(1)$ — there are 4 beads on each floor, of which 2 are visible.

The two ranges are recall bounds on an enumeration, not the extent of the phenomenon; §6.3 removes them.

Theorem 6.5 (the accounting law). *On both floors, and for every bead of Observation 6.4,*

$$k \text{ odd} \iff \text{the bead is visible} \iff N(\varepsilon^k) = -1. \tag{6.15}$$

Proof of the second equivalence. On the norm-(-1) side $N(\varepsilon_D) = -1$, so

$$N(\varepsilon_D^k) \;=\; (-1)^k \tag{6.16}$$

and the right-hand equivalence of (6.15) is the parity of k restated. ∎

The content of (6.15) is therefore the left-hand equivalence, which ties a group-theoretic property of the bead to the parity of its index; Theorem 6.8 below supplies its mechanism, in the form of the two Pell equations that separate the cosets. The three columns it identifies are these:

k	ε_8^k	φ^k	norm	parity	bead
1	$1+\sqrt{2}$	φ	-1	odd	visible
2	$3+2\sqrt{2}$	$(3+\sqrt{5})/2$	$+1$	even	invisible
3	$7+5\sqrt{2}$	$2+\sqrt{5}$	-1	odd	visible
4	$17+12\sqrt{2}$	$(7+3\sqrt{5})/2$	$+1$	even	invisible
5	$41+29\sqrt{2}$	$(11+5\sqrt{5})/2$	-1	odd	visible

Remark 6.6 (the order pattern differs; the visibility pattern does not). The two necklaces agree on everything the modular group can measure and disagree on everything else:

	$\Gamma_0(2)^+$, disc 8	$\Gamma_0(5)^+$, disc 5
bead spacing	$\operatorname{Reg}\mathbb{Q}(\sqrt{2}) = \ln(1+\sqrt{2})$	$\operatorname{Reg}\mathbb{Q}(\sqrt{5}) = \ln\varphi$
bead positions	$k\operatorname{Reg}$, $u = \varepsilon_8^k$	$k\operatorname{Reg}$, $u = \varphi^k$
odd k	order 4; stabiliser meets $\Gamma_0(2)$	order 2; stabiliser in $\Gamma_0(5)$
even k	order 2; stabiliser wholly in the Fricke coset	order 2; stabiliser wholly in the Fricke coset
order pattern	alternating 4/2	uniform 2
visibility pattern	alternating	alternating
beads per $X(1)$ period $4\operatorname{Reg}$	4	4
pivots seen from $X(1)$	2	2

At disc 8 the visible bead is an order-4 point which $\mathrm{PSL}(2,\mathbb{Z})$ reaches only through its square, an order-2 element of $\Gamma_0(2)$; at disc 5 it is an order-2 point outright. Either way what $\mathrm{PSL}(2,\mathbb{Z})$ sees is an order-2 pivot, which is why a view from $X(1)$ alone cannot separate the two discs at the pivot layer. The order itself is set by the first Fricke rung and hence by the single inequality of §4: the odd coset of $\Gamma_0(2)^+$ contains the trace $\sqrt{2} = 2\cos(\pi/4) < 2$, an order-4 element, while the smallest nonzero odd-coset trace of $\Gamma_0(5)^+$ is $\sqrt{5} > 2$ and already hyperbolic, so its only elliptic odd-coset trace is 0 — order 2.

Proposition 6.7 (the pivots, counted without the beads). *Enumerating instead the involutions of* $\mathrm{PSL}(2,\mathbb{Z})$ *whose fixed point lies on the axis returns, with the last column counting per period* $4\operatorname{Reg}$ *of the geodesic on* $X(1)$*,*

disc	pivots on the axis (k)	spacing	per period
8	$-5, -3, -1, 1, 3, 5$	2	2
5	$-7, -5, -3, -1, 1, 3, 5, 7, 9$	2	2

— two pivots per period on both floors, and the pivots are precisely the odd-k beads.

Verification. The computation shares no step with the bead analysis; the two agree [App. A]. The disc-8 pivots come out at the coordinates $u = \pm 1 + \sqrt{2}$, $\pm 7 + 5\sqrt{2}$, $\pm 41 + 29\sqrt{2}$. On the $t = 6$ axis the same passages had been computed from the other side and by a different route: a point $(a+i)/c$ of the orbit of i lies on that axis exactly when $(a-c)^2 - 2c^2 = -1$, and a sweep over $c \leq 10^5$ returns $(a,c) = (0,1), (2,1), (-2,5), (12,5), (-12,29), (70,29), \ldots$, i.e. the coefficient pairs $(|a-c|, c) = (1,1), (7,5), (41,29)$ — the first three solutions of the negative Pell equation of discriminant 8. The same three pairs, in the same order, from two computations that share no step [App. A]. Two pivots per period is also the general statement of §7, obtained there from the structure of the inverting involutions rather than from an enumeration.

6.3 The ladder is a Pell ladder

Observation 6.4 is bounded in k; the following removes the bound at the two floors where its hypotheses hold.

Theorem 6.8 (the bead ladder). *Let $p \in \{2, 5\}$, let N be the floor element of $\Gamma_0(p)^+$ and A its axis. Then the stabiliser of A in $\Gamma_0(p)^+$ is infinite dihedral,*

$$\mathrm{Stab}(A) \;=\; \langle N \rangle \rtimes \langle \sigma \rangle \;\cong\; D_\infty, \tag{6.17}$$

and its reflection set — equivalently, by Leg 2 below, the set of beads on A — is exactly

$$\{\varepsilon^k \;:\; k \in \mathbb{Z}\} \tag{6.18}$$

in the coordinate (6.14), for all k, where $\varepsilon = \varepsilon_8$ at $p = 2$ and $\varepsilon = \varphi$ at $p = 5$.

Proof, in three legs.

Leg 1 (N is primitive on its axis). Stab(A) is a discrete group of isometries of a line, hence infinite cyclic or infinite dihedral. If some $M \in \Gamma_0(p)^+$ with axis A satisfied $M^j = N$ for some $j \geq 2$, then M would be hyperbolic of translation length $\ell(N)/j$, hence of trace strictly inside

$$\left(2,\; n\sqrt{p}\right) \;=\; \begin{cases} (2,\; 2\sqrt{2}) = (2,\; 2.828427125\ldots), & p = 2,\; n = 2, \\ (2,\; \sqrt{5}) = (2,\; 2.236067977\ldots), & p = 5,\; n = 1. \end{cases} \tag{6.19}$$

By §3.1 the trace of an element of $\Gamma_0(p)^+$ lies in $\mathbb{Z}$ or in $\sqrt{p} \cdot \mathbb{Z}$ according to its coset. Neither interval in (6.19) contains an integer, and neither contains a lower hyperbolic rung: at $p = 5$ the floor element is the first rung, and at $p = 2$ the first rung has trace $\sqrt{2} < 2$ and is elliptic. So no such M exists and N is primitive. This leg is exactly the hypothesis that the floor element is the floor's systole, which by Remark 3.15 holds at $D = 5$ and $D = 8$ — that is, at $p = 5$ and $p = 2$.

Leg 2 (every elliptic order is even). A cone point of order m on A contributes a half-turn to Stab(A) if and only if m is even, the rotation by π being the $(m/2)$-th power. Were some order odd, a cone point could sit on A without being a reflection of the stabiliser and the identification of beads with reflections would fail. The orders are therefore enumerated rather than assumed, from the realised elliptic traces of each coset: at $p = 2$ these are 0 in the even coset and $0, \pm\sqrt{2}$ in the odd coset, giving cone orders $\{2, 4\}$; at $p = 5$ they are 0 in each coset, giving $\{2\}$. Both sets consist of even orders. (The signatures $(0; 2, 4; 1)$ and $(0; 2, 2, 2; 1)$ are proved in §5; this leg re-derives only the parity, and re-derives it from the trace sets, so the two are independent.)

Leg 3 (the on-axis condition is a Pell equation). For an involution of $\Gamma_0(p)^+$, the condition that its fixed point lie on A reduces, coset by coset, to an exact Pell-type equation in the matrix entries, so that the search over the stabiliser is a one-parameter loop rather than a box:

p	coset	on-axis condition	coordinate u of the fixed point
5	even	$C^2 - 5aC + 5a^2 = -5$	$(C/2 - a) + (C/10)\sqrt{5}$
5	odd	$c^2 - 5ac + 5a^2 = -1$	$c/2 + (c/2 - a)\sqrt{5}$

p	coset	on-axis condition	coordinate u of the fixed point
2	even	$C^2 - 2a^2 = 2$	$-a + (C/2)\sqrt{2}$
2	odd	$c^2 - 2a^2 = 1$	$c - a\sqrt{2}$

The bead ladder is a Pell ladder, and that is the reason the answer is a power of the fundamental unit rather than an arbitrary algebraic number: the right-hand sides are the norm-form values that index the odd and the even powers of ε, so the coset of an involution — and with it the visibility of its bead — is decided by the parity of the index. This is Theorem 6.5's left-hand equivalence, arriving from underneath.

Assembly. By Leg 1, Stab(A) contains the primitive hyperbolic N; it also contains at least one half-turn, namely w_p itself, which is the $k = 0$ bead of Observation 6.4. Hence (6.17). In D_∞ the involutions are exactly σN^k, and the fixed point of σN^k sits at arclength

$$s_k \;=\; s_0 + k \cdot \frac{\ell(N)}{2} \;=\; k \log \varepsilon \tag{6.20}$$

using $s_0 = 0$ and $\ell(N) = 2 \log \varepsilon$ — Corollary 3.5's floor eigenvalue $\lambda_N = \varepsilon_D$ read as a length, and (6.5) at $p = 2$ — so that by (6.14)

$$u \;=\; e^{s_k} \;=\; \varepsilon^k. \tag{6.21}$$

Leg 2 says every bead on A is such an involution, and (6.18) follows. ■

Verification. The Pell reduction of Leg 3 makes the enumeration exhaustive in a parameter rather than in a box: over $|a| \le 4000$, exhaustively, it returns 36 axis involutions at $p = 5$ with 35 distinct coordinates u, all of them powers of ε and covering 35 consecutive k with $-16 \le k \le 18$; and 21 axis involutions at $p = 2$ with 21 distinct coordinates, all powers of ε, covering 21 consecutive k with $-10 \le k \le 10$ [App. A]. These enumerations are controls on the theorem rather than its evidence: what is proved is (6.18) for all k, and what is observed is the stated range. The earlier and independent enumeration of Observation 6.4, run from a different parametrisation with a search box of ± 60, returned 19 and 11 consecutive k respectively and agrees with these on the overlap [App. A].

6.4 Where the ladder does not close

Theorem 6.8 is stated at $p = 2$ and $p = 5$, and the restriction is not an artefact of what was computed. Each of Legs 1 and 2 carries a hypothesis, and neither is generic.

Remark 6.9 (the two hypotheses, and the scope they leave). Leg 1 needs the floor element to be the systole of its floor. Over the population of Theorem 3.3 — the 28 fundamental discriminants D with $5 \le D \le 204$ and $N(\varepsilon_D) = -1$ — that holds for exactly $D \in \{5, 8, 13, 53\}$ and fails first at $D = 17$, by Remark 3.15. Leg 2 needs every elliptic order of the floor to be even. Accordingly the bead law is a statement about a Fricke floor *whose floor element is its systole*: there the systolic axis carries beads at exactly the positions $k\,\mathrm{Reg}$, $k \in \mathbb{Z}$, indexed by the powers ε^k; and where the hypothesis fails, the failure is not that the law is false but that its subject is absent — there is no ε^k ladder on the systolic axis for the law to describe, because the element whose eigenvalue is ε is not the systole. The two floors of Theorem 6.8 are those of $D = 8$ and $D = 5$, both in the four-member set, and both hypotheses hold there.

Whether a given floor satisfies the first hypothesis is decidable, and by a test that is finite in an explicit range.

Proposition 6.10 (the floor-is-systole criterion). *For squarefree p put*

$$n_0(p) \ = \ \min\{\, k \geq 1 \ : \ k^2 p > 4 \,\}, \tag{6.22}$$

$$t_{\min}(p) \ = \ \min\{\, t \geq 3 \ : \ t \text{ is a trace of } \Gamma_0(p) \,\}. \tag{6.23}$$

The systole trace of $\Gamma_0(p)^+$ is the smaller of the two progressions above the parabolic threshold,

$$\min\bigl(t_{\min}(p),\ n_0(p)\sqrt{p}\bigr), \tag{6.24}$$

and consequently the rung n is the shortest closed geodesic of $\Gamma_0(p)^+$ if and only if

$$n \ = \ n_0(p) \qquad \text{and} \qquad n_0(p)\sqrt{p} \ < \ t_{\min}(p). \tag{6.25}$$

The criterion is effective: only the traces $t \in \{3, \dots, \lfloor n_0\sqrt{p}\rfloor\}$ need testing, an explicit finite range, and the test runs in $O(\sqrt{p} \cdot \omega(p) \cdot p)$.

Verification. On the 28-member population the criterion has 4 true positives, 0 false positives, 0 false negatives and 24 true negatives, the four positives being $D \in \{5, 8, 13, 53\}$ [App. A]. Its two clauses separate the two failure mechanisms of Remark 3.15: $n = n_0$ is the first, $n_0\sqrt{p} < t_{\min}$ the second. The three discriminants that a rung-geometry test alone admits are each rejected by the second clause — $D = 29$, where $\Gamma_0(29)$ has a trace-3 element because 5 is a quadratic residue mod 29, against $\sqrt{29} = 5.3852\dots$; $D = 85$, where the integer trace 5 beats $\sqrt{85} = 9.2195\dots$; and $D = 173$, where 5 beats $\sqrt{173} = 13.1529\dots$ A second control, free to disagree with the first, ran the criterion over every squarefree p with $2 \leq p \leq 20{,}000$, exhaustively, and returned exactly 25 levels,

$$p \in \{2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 21, 30, 34, 38, 42, 53, 58, 65, 130, 186, 246, 370, 390, 966, 2046\}, \tag{6.26}$$

the largest being 2046 — reproducing a list obtained independently by a direct level scan [App. A]. That the list is complete over all p is *not* claimed here; §11 names what a proof would need.

Remark 6.11 (a re-scoping condition that does no work). It is tempting to cut the population by asking that $n^2p - 4$ be a perfect square. The identity (3.11) makes the condition vacuous:

$$n^2 p - 4 \ = \ v^2 D - 4 \ = \ u^2 \tag{6.27}$$

holds for every member of the 28-member population, $D = 17$ included — that is, including precisely the case such a condition would be introduced to exclude. The hypothesis that does the work is the one in Proposition 6.10.

7. Reciprocity and threading

On the modular surface a geometric condition and an algebraic one name the same closed geodesics. The geometric condition is that the geodesic passes through the order-2 cone point; the algebraic one is that the class is equal to its own inverse. §7.1 shows that these two are equivalent, together with two further descriptions of the same set, and counts the passages — exactly two per period. §7.2 then replaces the existence question at a given trace by a single Diophantine criterion, and locates the two families of traces produced by the earlier sections inside it: they are conditions under which that one criterion is met, not mechanisms competing with it. §7.3 measures the distance between reciprocity and the norm of a fundamental unit, which is not zero. §7.4 carries the whole of it up to the Hecke stages, where it survives intact at $G(4)$ and fails at $G(5)$ in a way that is itself a result, and that is where this section's domain ends.

7.1 Reciprocity on the modular surface

Throughout this section

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.1}$$

is the elliptic element of order 2 of $\mathrm{PSL}(2,\mathbb{Z})$ fixing the point i, which projects to the order-2 cone point of $X(1)$. A hyperbolic class $[M]$ is called *reciprocal* when $[M] = [M^{-1}]$, and it *threads the orbit of* i when the axis of one — equivalently of every — representative meets $\mathrm{PSL}(2,\mathbb{Z}) \cdot i$.

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1,

$$SMS^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{7.2}$$

so that, as an identity of matrices,

$$SMS^{-1} = M^{-1} \qquad \Longleftrightarrow \qquad b = c. \tag{7.3}$$

In $\mathrm{PSL}(2,\mathbb{Z})$ one must also allow $SMS^{-1} = -M^{-1}$; that forces $a = d = 0$ and $b = -c$, hence $M = \pm S$, which is elliptic, so for hyperbolic M the two readings of (7.3) coincide. The fixed points of M solve

$$c\,z^2 + (d-a)\,z - b = 0, \tag{7.4}$$

whose root product is $-b/c$; therefore

$$b = c \qquad \Longleftrightarrow \qquad z_+ z_- = -1, \tag{7.5}$$

and $z_+ z_- = -1$ says precisely that the endpoint pair is preserved by $z \mapsto -1/z$, that is, that the axis passes through i.

Theorem 7.1 (the reciprocity equivalences). *For a hyperbolic class* $[M]$ *of* $\mathrm{PSL}(2,\mathbb{Z})$ *the following are equivalent:*

(i) the axis of some — hence of every — representative passes through a point of the orbit of i*;*

(ii) $[M] = [M^{-1}]$*;*

(iii) $[M]$ *contains a symmetric matrix;*

(iv) some representative is inverted by an involution of the group whose fixed point lies on its axis.

Proof.

Step 1 (symmetry and the axis). The equivalence of (iii) with the special case of (i) in which the axis passes through i itself is (7.3) together with (7.5).

Step 2 ((i) ⇒ (ii) and (iv)). Suppose the axis of M passes through $\gamma \cdot i$ for some $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$. Then $N = \gamma^{-1}M\gamma$ has axis through i, so $SNS^{-1} = N^{-1}$ by Step 1, and with $J = \gamma S\gamma^{-1}$ — an element of order 2 fixing $\gamma \cdot i$ —

$$JMJ^{-1} \;=\; M^{-1}. \tag{7.6}$$

Step 3 ((ii) ⇒ (i)). Suppose $gMg^{-1} = M^{-1}$ with M hyperbolic. Then g maps the endpoint pair of M to itself. It cannot fix both endpoints: it would then preserve the axis with its orientation and commute with the endpoint stabilizer, giving $M^{-1} = gMg^{-1} = M$, impossible for hyperbolic M. So g swaps the endpoints and reverses the axis; restricted to the axis it is an orientation-reversing isometry of a line and has a fixed point p there. Then g^2 fixes both endpoints and the interior point p, so $g^2 = 1$. The same computation runs through the whole inverting family, since

$$(M^k g)^2 \;=\; M^k\big(gM^k g^{-1}\big)g^2 \;=\; M^k M^{-k} \;=\; 1 \tag{7.7}$$

for every k: *every* element conjugating M to M^{-1} is an involution with fixed point on the axis. Finally $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ [15], and torsion in a free product is conjugate into a free factor [13], so every element of order 2 is conjugate to S; hence $p \in \mathrm{PSL}(2,\mathbb{Z}) \cdot i$ and the axis threads the orbit of i. ■

Verification. The identity (7.2) and the equivalence (7.3) were checked in exact integer arithmetic on 306 sampled matrices of $\mathrm{SL}(2,\mathbb{Z})$ — 30 symmetric, 276 not — with 0 failures [App. A].

The involutions of Step 3 are the *pivots* of the class, and they are not numerous.

Proposition 7.2 (two pivots per period). *Let* $[M]$ *be a primitive reciprocal class. The elements inverting* M *are exactly the* $M^k g$*,* $k \in \mathbb{Z}$*, for any one of them* g*, and*

$$M\,(M^k g)\,M^{-1} \;=\; M^{k+2}\,g, \tag{7.8}$$

so their fixed points fall into exactly two classes modulo the translation M*: a primitive reciprocal geodesic passes through exactly two points of the orbit of* i *per period.*

Verification. Two distinct pivots were exhibited on every reciprocal class of trace $t \le 40$; over the same range, the number of representations of $t^2 - 4$ in the form of (7.13) below was in every case exactly twice the number of reciprocal classes at that trace [App. A].

Example 7.3 (the two half-turns at trace 6). The symmetric representative

$$M = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \operatorname{tr} M = 6, \tag{7.9}$$

is inverted by the two involutions

$$\sigma_1 = S \text{ (fixing } i), \qquad \sigma_2 = T^2 S T^{-2} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \text{ (trace 0, fixing } 2+i), \tag{7.10}$$

and M is their product:

$$\sigma_2 \sigma_1 = \begin{pmatrix} -5 & -2 \\ -2 & -1 \end{pmatrix} = -M = M \text{ in } \mathrm{PSL}(2, \mathbb{Z}). \tag{7.11}$$

The two pivots of Proposition 7.2 are here i and $2+i$, and the class is the product of the half-turns about them.

7.2 The one criterion

Theorem 7.1 turns the question "which classes thread" into a question about symmetric matrices, and that question is Diophantine. A symmetric $M = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ of trace t and determinant 1 exists exactly when $a(t-a) - b^2 = 1$, that is, when

$$(2a - t)^2 + (2b)^2 = t^2 - 4. \tag{7.12}$$

Since $t^2 - 4 \equiv 0$ or $1 \pmod 4$ and never 2 or 3, the even leg of any representation of $t^2 - 4$ as a sum of two squares may be taken to be $2b$. This gives the criterion, in the form it will be used in for the rest of the section.

Theorem 7.4 (the threading criterion). *Let $t \ge 3$ be an integer. Some class of trace t threads the orbit of i if and only if*

$$t^2 - 4 = s^2 + (2b)^2, \qquad b \ne 0, \tag{7.13}$$

is solvable in integers; equivalently, writing the condition by parity,

$$t \text{ odd}: \ t^2 - 4 \ \text{ a sum of two squares}; \qquad t \text{ even}: \ \tfrac{1}{4}(t^2 - 4) \ \text{ a sum of two squares}; \tag{7.14}$$

equivalently again, by the two-squares theorem [16], no prime $\equiv 3 \pmod 4$ *divides the relevant quantity to an odd power.*

Proof. Given a solution of (7.13), the matrix $\left(\begin{smallmatrix} (t+s)/2 & b \\ b & (t-s)/2 \end{smallmatrix}\right)$ has determinant 1 and trace t; parity is automatic because $s^2 \equiv t^2 \pmod 4$ forces $s \equiv t \pmod 2$. Conversely a symmetric representative gives (7.12). The parity refinement (7.14) is the observation that for odd t any two-square representation of the odd number $t^2 - 4$ has exactly one even leg, and that for even t the condition $4 \mid t^2 - 4$ forces s even. ■

Verification. The equivalence of the exhaustive search over (s, b) with the factorisation criterion was confirmed for every integer trace $3 \le t \le 40$, exhaustively in (s, b) at each trace [App. A]. The criterion was obtained independently by a second derivation, which reaches (7.12) from (7.3) and (7.5) and no other input [App. A].

The traces produced by the two constructions of §3 both satisfy the criterion, and the first of them does so for a reason that can be written on one line.

Theorem 7.5 (every Fricke-floor trace threads). *For $t = u^2 + 2$ with $u \ge 1$ — the traces of the norm-(-1) family of Theorem 3.3 —*

$$t^2 - 4 \;=\; (t-2)(t+2) \;=\; u^2\big(u^2+4\big) \;=\; (u^2)^2 + (2u)^2, \tag{7.15}$$

so (7.13) is solvable with witness $(s, b) = (u^2, u) = (t-2, \operatorname{Tr}\varepsilon_D)$, and every such trace threads. The pivot arithmetic and the floor arithmetic are the same two numbers.

Verification. Checked in exact arithmetic for $u = 1, \ldots, 12$; the predicted threading set $\{3, 6, 11, 18, 27, 38\}$ is contained in the threading set $\{3, 6, 7, 11, 15, 18, 27, 34, 38, 39\}$ computed independently over $3 \le t \le 40$ [App. A].

The second construction of §3 — the degenerate branch $p = 1$ of Remark 3.2, where $t + 2 = n^2$ and the trace-t class is already a square inside $\mathrm{PSL}(2, \mathbb{Z})$ — stands in a different relation to the criterion. There

$$t^2 - 4 \;=\; (t-2)\, n^2, \tag{7.16}$$

a square factor times $t - 2$. At $t = 7$ and $t = 34$ the criterion is met, and the threading there is *only* of this kind: the threading classes are the squares of the trace-3 and trace-6 classes, produced here by the level-selection identity (3.11) and identified independently, at word level, as $(LR)^2$ and $(L^2R^2)^2$ — the trace-3 and trace-6 geodesics traversed twice [App. A].

Neither family is a mechanism for threading. There is one criterion, Theorem 7.4, and each family is a way in which that criterion is already accounted for; the question of a *third mechanism* does not arise, because there is no first or second one for it to stand beside. What the two families do not do is exhaust the criterion.

Proposition 7.6 (the classification of threading traces). *Over $3 \le t \le 200$, exhaustively, there are 33 threading traces, distributed as follows.*

class	count	members
$t - 2$ a square (Theorem 3.3's family)	14	$3, 6, 11, 18, 27, 38, 51, \ldots$
$t + 2$ a square (the $p = 1$ branch)	4	$7, 34, 47, 119$

class	count	members
neither	**15**	**15**, **39**, 43, 63, 70, 87, ...

The class lying outside both families is the largest of the three: 15 *of* 33*, against the two families'* 18 *combined.*

Verification. The instrument was first run as a control over $3 \le t \le 40$, where it reproduced the independently computed threading set $\{3, 6, 7, 11, 15, 18, 27, 34, 38, 39\}$ exactly, before being run over $3 \le t \le 200$ [App. A]. Inside the shorter window $3 \le t \le 40$ the two families account for 8 of the 10 threading traces; the count in Proposition 7.6 is the count over $3 \le t \le 200$ and is not the same statement.

The two smallest traces outside both families are not anomalies, and what they are is visible in one line of Gaussian arithmetic.

Example 7.7 (the traces 15 and 39). At $t = 15$ and $t = 39$ the numbers $t - 2$ and $t + 2$ are themselves primes $\equiv 1 \pmod 4$, and neither is a perfect square:

t	$t-2$	$t+2$	t^2-4	witness (s, b)
15	$13 = 2^2 + 3^2$	$17 = 1^2 + 4^2$	$221 = 13 \cdot 17$	$(11, 5)$
39	$37 = 1^2 + 6^2$	$41 = 4^2 + 5^2$	$1517 = 37 \cdot 41$	$(29, 13)$

Their product is a sum of two squares by the Gaussian-integer identity, with no square anywhere. At $t = 15$:

$$13 \;=\; 2^2 + 3^2, \qquad 17 \;=\; 1^2 + 4^2, \tag{7.17}$$

$$(2 + 3i)(1 - 4i) \;=\; 14 - 5i, \qquad 14^2 + 5^2 \;=\; 221, \tag{7.18}$$

and the even coordinate is forced rather than lucky: 221 is odd, so exactly one coordinate is even in *every* representation.

7.3 Reciprocity and the norm of the fundamental unit

Threading is an arithmetic condition on $t^2 - 4$, and there is a second, older arithmetic condition on the same discriminant: whether the fundamental unit of the field it generates has norm -1. The two conditions track each other in one direction only.

Proposition 7.8 (the norm implies reciprocity; the converse is false). *Let $t \ge 3$ and let ε be the fundamental unit of $\mathbb{Q}(\sqrt{t^2 - 4})$.*

(a) If $N(\varepsilon) = -1$ then trace t is reciprocal. Indeed $N(\varepsilon) = -1$ forces every prime factor of the squarefree part of $t^2 - 4$ to be 2 or $\equiv 1 \pmod 4$, hence a sum of two squares, hence so is $t^2 - 4$.

(b) The converse fails. Over the machine scan $3 \le t \le 60$ the criterion of Theorem 7.4 holds with 0 *failures, while the equivalence with the solvability of the negative Pell*

equation fails at $t = 15$, $t = 39$ and $t = 43$. At the first of these,

$$221 \;=\; 13 \cdot 17 \;=\; 5^2 + 14^2, \tag{7.19}$$

so reciprocity holds and is realised by

$$M \;=\; \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}, \qquad \det M \;=\; 26 - 25 \;=\; 1, \tag{7.20}$$

while

$$\varepsilon_{221} \;=\; \frac{15 + \sqrt{221}}{2}, \qquad N(\varepsilon_{221}) \;=\; \frac{225 - 221}{4} \;=\; +1, \tag{7.21}$$

so $x^2 - 221\,y^2 = -1$ is not solvable. Both prime factors of 221 are $\equiv 1 \pmod 4$, so the classical necessary condition for $N(\varepsilon) = -1$ holds here and is not sufficient.

The honest law is therefore that reciprocity is a sum-of-two-squares condition, strictly weaker than norm -1. The trace $t = 15$ of Example 7.7 is the trace at which the two conditions first come apart.

7.4 The boundary: the Hecke stages

Everything above is a statement about $X(1)$. It survives one step next door, and the exact manner of its failure at the following step is the last result of this section.

Proposition 7.9 (the instance at $G(4)$). *Let $A = TST$ be the minimal hyperbolic element of $G(4)$, of trace $2\sqrt{2}$. Every clause of §7.1 holds for it, with the group's own elements as witnesses.*

(i) Inversion is inner, realised by the elliptic S:

$$S\,(TST)\,S^{-1} \;=\; TST^{-1}. \tag{7.22}$$

(ii) A is a product of two half-turns,

$$TST \;=\; (TS)^2 \cdot S, \qquad (TS)^2 \;=\; \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \tag{7.23}$$

where S is the half-turn at i and $(TS)^2$ the half-turn at the order-4 cone point $z_0 = e^{i\pi/4}$ — the involution contributed by that point's cyclic stabilizer of order 4.

(iii) Consequently the axis is forced to be the geodesic through the two cone points, and the translation length is twice their separation:

$$\cosh d(i, z_0) \;=\; 1 + \frac{2 - \sqrt{2}}{\sqrt{2}} \;=\; \sqrt{2}, \tag{7.24}$$

so that

$$\ell \;=\; 2\,d(i, z_0) \;=\; 2\ln\!\big(1 + \sqrt{2}\big) \;=\; 1.762747174. \tag{7.25}$$

(iv) The axis carries infinitely many cone points, strictly alternating in order and evenly spaced: with u *the unit-speed coordinate along the axis and* $d = d(i, z_0)$*,*

$$\frac{u}{d} \in \{\dots, -3, -1, 1, 3, \dots\} \text{ (order 4)}, \qquad \frac{u}{d} \in \{\dots, -2, 0, 2, \dots\} \text{ (order 2)}, \tag{7.26}$$

and the stabilizer of the axis in $G(4)$ *is the infinite dihedral group generated by the half-turns at consecutive points, with* $\langle A \rangle$ *its index-*2 *translation subgroup — two passages per period, as in Proposition 7.2.*

(v) The class contains a symmetric matrix, and $q = 4$ *is the only Hecke order at which the minimal element does: by (5.36) the matrix* TST *has equal off-diagonal entries exactly when* $\lambda^2 - 1 = 1$*, so it is symmetric if and only if* $\lambda^2 = 2$*, that is if and only if* $q = 4$ *— the condition (5.37) that also places the home cone points on the axis, reached here by a different route. Clause (iii) of Theorem 7.1 is thereby instantiated at* $G(4)$ *as well.*

Verification. Parts (i)–(iii) are exact identities in $\mathbb{Z}[\sqrt{2}]$, the length in (7.25) matching $2\,\mathrm{arccosh}\sqrt{2}$ to residual 4.4×10^{-16}. Part (iv) is bounded: all 6,268 distinct elements of the ball of word length ≤ 12 were applied to z_0 and to i, yielding 3,135 distinct order-4 orbit points and 4,061 order-2 orbit points, and the pattern (7.26) held to six decimals throughout; the 18 minimal-trace elements of word length ≤ 8 were shown conjugate in $G(4)$ to A or A^{-1} with the conjugators exhibited [App. A].

At $G(5)$ the equivalence of Theorem 7.1 still holds class by class, and the shortest closed geodesic is not one of the classes it selects.

Proposition 7.10 (the class census at $G(5)$**).** *Write* $G(5) = \mathbb{Z}/2 * \mathbb{Z}/5$ *with generators* S *of order* 2 *and* U *of order* 5*; conjugacy classes of hyperbolic elements are cyclic words* $SU^{a_1} \cdots SU^{a_m}$ *with* $a_i \in \{1, 2, 3, 4\}$*, taken up to rotation, and the inverse class is the reversed sequence with* $a_i \mapsto 5 - a_i$ *— so reciprocity is decidable exactly, with no search. Over the* 32 *classes of syllable length* $m \leq 3$*:*

cyclic word	trace	reciprocal	threads
SU^2 (the systole, §5)	2φ	no	no
SU^3	2φ	no	no
SU^1SU^4	$3 + \varphi$	**yes**	**yes**
SU^2SU^3	$4 + 3\varphi$	**yes**	**yes**
the other 28 classes	—	no	no

Two classes are reciprocal, two thread, and they are the same two. The reciprocal words are exactly the complementary palindromes,

$$a_i + a_{m+1-i} \;=\; 5, \tag{7.27}$$

and the shortest closed geodesic is not one of them: at $q = 4$ *it is the equivalence's flagship instance (Proposition 7.9), at* $q = 5$ *it lies outside the equivalence entirely, which is carried instead by two longer classes.*

Verification. Reciprocity is decided exactly from the word. Threading was tested at 40 digits against every elliptic fixed point in the ball, with threshold 10^{-20}; the negative half of the table is therefore recall-bounded over that ball, while the reciprocity column is not [App. A].

> **Remark 7.11 (why the pivots run out).** The mechanism is the parity of q, not commensurability. On $X(1)$ and on $G(4)$ a reciprocal element is a product of two half-turns at cone points on its axis, and a half-turn exists at a cone point of order m only when m is even, being the $(m/2)$-th power of the generator there. The cone orders of $G(5)$ are 2 and 5; at the order-5 point there is no half-turn. The golden stage has only its order-2 orbit to offer as pivots, and the shortest geodesic's axis misses it.

What does not survive the step is the arithmetic partner of reciprocity. On $X(1)$ the norm of the fundamental unit implies reciprocity and is not implied by it (Proposition 7.8). At $G(5)$ the corresponding unit-level statement fails in both directions. Write $K = \mathbb{Q}(\sqrt{5})$, let σ be its nontrivial automorphism, and for a hyperbolic class of trace $t \in \mathbb{Z}[\varphi]$ let $L = K[\varepsilon]$ be the field generated by its eigenvalue and $\mathcal{O}_K[\varepsilon]$ the order the class attaches to. Call the class *reciprocal in* $G(5)$ when it is inverted by an element of $G(5)$ itself.

> **Proposition 7.12 (the order-level biconditional fails in both directions).** *Neither implication holds between "the order $\mathcal{O}_K[\varepsilon]$ has a unit of relative norm -1" and "the class is reciprocal in $G(5)$". Each direction is refuted by an explicit witness, and the two witnesses were produced by computations independent of one another.*
>
> *(i) A unit of relative norm -1 present, class not reciprocal — at $t = 4\varphi$. The unit is exhibited, with coordinates $(-2-\varphi, -1+\varphi)$ and relative norm exactly -1; the class is the cyclic word $(1, 1, 2)$, whose reverse-complement in the sense of Proposition 7.10 is*
>
> $$(1, 1, 2) \longmapsto (3, 4, 4) \neq (1, 1, 2), \tag{7.28}$$
>
> *and no inverting conjugator exists in a ball of 152,914 elements of PSL_2, all reduced words of syllable length ≤ 15.*
>
> *(ii) Class reciprocal, no unit of relative norm -1 — at $t = 43 + 65\varphi$, the class $(1, 3, 1, 4, 2, 4)$. It is reciprocal, inverted by S, with the conjugator exhibited; and its order provably has no unit of relative norm -1, by a congruence at a prime above 11. This half is a proof, not a search.*

What survives both computations is not the unit-level statement but an archimedean one, and it survives with a mechanism at each end.

> **Theorem 7.13 (the archimedean implication).** *Let W be a hyperbolic element of $G(5)$ of trace t, and let t^σ denote the trace read at the second real place of K. If W is reciprocal in $G(5)$, then*
>
> $$|t^\sigma| > 2. \tag{7.29}$$

Proof. Reciprocity supplies an involution ι with $\iota W \iota^{-1} = W^{-1}$, so that

$$W = (W\iota) \cdot \iota \tag{7.30}$$

exhibits W as a product of two π-rotations. Both factors have trace 0, and a trace-0 element has trace 0 in every real embedding; so the images of both under σ are again π-rotations of $\mathbb{H}$. A product of two π-rotations about distinct points is hyperbolic with $|\text{trace}| = 2\cosh d$, d the distance between the centres, and this exceeds 2. ■

Verification. The implication holds at all five of the classes registered in advance, and over 3,360 cyclic classes of syllable length $n \leq 7$ — of which 3,346 are hyperbolic and 42 are reciprocal — all 42 satisfy (7.29), with 0 violations [App. A].

The converse of Theorem 7.13 is false, and it is refuted by the same class that refutes half of Proposition 7.12: at $t = 4\varphi$ one has $t^\sigma = -2.472135955$, so $|t^\sigma| > 2$ and L is totally real, while the class is not reciprocal. Two computations reached this counterexample independently of one another.

Remark 7.14 (one instrument, two filters). The two computations organised themselves around different quantities — one around t^σ, the other around $\sigma(t^2 - 4)$ — and the quantities are the same instrument:

$$|t^\sigma| > 2 \quad \Longleftrightarrow \quad \sigma(t^2 - 4) > 0 \quad \Longleftrightarrow \quad L \text{ totally real}, \tag{7.31}$$

verified as an identity across all five registered classes [App. A].

This is where the section's domain ends, and it ends at a stated place rather than a vague one. On $X(1)$ the four descriptions of Theorem 7.1 are one description, the existence question at each trace is settled by the single criterion of Theorem 7.4, and the norm of the fundamental unit is a sufficient but not necessary condition for it. On $G(4)$ every clause is instantiated inside the group, half-turns and all. On $G(5)$ the equivalence itself survives — reciprocity and threading still name the same classes — but the arithmetic partner changes: the unit-level statement is false in both directions, and the invariant that co-tracks reciprocity is archimedean rather than unit-theoretic, in the one direction (7.29) and not the other. The mechanism of the order-level failures, and what the surviving implication does not decide, are taken up in §11.

8. The realization laws, and the trace gaps

A group's traces are drawn from a ring, and only some of that ring's elements are traces. This section says which. On $X(1)$ the answer is *all of them*: every integer is the trace of an element of the modular group, and at each hyperbolic integer the conjugacy classes are exhibited. On $G(4)$ the answer is a closed form, $2\mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$, whose two progressions are graded by the two cosets of $\Gamma_0(2)$ in $\Gamma_0(2)^+$ everywhere except at the single value where they meet. What is left over — the odd integers, the mixed elements $a + b\sqrt{2}$ — is not rare in the group, or long, or deep: it is absent, and §8.5 says in what sense the absence is sharp. The last subsection leaves the individual group and asks which real numbers are traces of nothing whatever; there the two sides of the seam answer differently, and the difference is a theorem with an exhibited witness.

8.1 Realized traces, and the two ambient objects

For a subgroup $G \leq \mathrm{PSL}(2, \mathbb{R})$ write

$$\Theta(G) \;=\; \{\, \operatorname{tr} g \;:\; g \in G \,\}, \tag{8.1}$$

the *realized trace set*, understood up to sign throughout (the trace of an element of $\mathrm{PSL}(2, \mathbb{R})$ is defined only up to sign). For the stages $G(q)$ of §1, with $\lambda_q = 2\cos(\pi/q)$, every trace is an algebraic integer of the *ambient trace ring*

$$\mathbb{Z}[\lambda_q] \;\supseteq\; \Theta(G(q)), \tag{8.2}$$

so that $\mathbb{Z}[\lambda_q]$ is $\mathbb{Z}$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\varphi]$ and $\mathbb{Z}[\sqrt{3}]$ at $q = 3, 4, 5, 6$ respectively. Two ambient objects are in play and they must not be confused: the ring (8.2), which is **countable**, and the *ambient trace interval*

$$(-2, 2) \quad \text{(elliptic)}, \qquad |t| = 2 \quad \text{(parabolic)}, \qquad (2, \infty) \quad \text{(hyperbolic)}, \tag{8.3}$$

which is a **continuum**. A value of the ring or of the interval that is not realized will be called a *trace gap* of G; this and *realized trace set* are the section's only coinages.

The two sides of (8.3) behave differently before any group is fixed. On the elliptic side the realized set is determined at every stage by the signature alone: the cone orders of $(2, q, \infty)$ are 2 and q, and the elliptic realized set is

$$\left\{ 2\cos(k\pi/m) \;:\; m \in \{2, q\},\; 1 \leq k < m \right\} \;=\; \{0\} \cup \left\{ 2\cos(k\pi/q) \;:\; 1 \leq k < q \right\}, \tag{8.4}$$

which evaluates to $\{0, \pm 1\}$ at $q = 3$, to $\{0, \pm\sqrt{2}\}$ at $q = 4$, to $\{0, \pm\varphi, \pm\varphi^{-1}\}$ at $q = 5$ and to $\{0, \pm 1, \pm\sqrt{3}\}$ at $q = 6$. On the hyperbolic side nothing of the kind holds: the realized set is not read off the signature, and it is given in closed form below at two of the four stages, §11 recording what is open at a third.

8.2 The realized set of $X(1)$

Theorem 8.1 (every integer is a trace). *Every element of* $\mathrm{PSL}(2,\mathbb{Z})$ *has integer trace, and every integer is realized:*

$$\Theta\big(\mathrm{PSL}(2,\mathbb{Z})\big) \;=\; \mathbb{Z}. \tag{8.5}$$

Split by type, the elliptic part is $\{0,\pm1\}$*, the parabolic value* ± 2 *is realized by* T*, and the hyperbolic part is all of* $\{t \in \mathbb{Z} : t \geq 3\}$ *— the realization on the hyperbolic side is complete.*

Proof. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with integer entries,

$$\operatorname{tr} M \;=\; a + d \;\in\; \mathbb{Z}, \tag{8.6}$$

which gives $\Theta \subseteq \mathbb{Z}$. Ellipticity is $|t| < 2$, so the elliptic traces are exactly $|t| \in \{0,1\}$, realized by S and U; the parabolic value $|t| = 2$ is realized by T (§10); and for every integer $t \geq 3$ the construction of Proposition 3.1 exhibits an element $N^2 \in \Gamma_0(p) \subset \mathrm{PSL}(2,\mathbb{Z})$ of trace t. ■

Verification. The hyperbolic realization was carried out in exact integer arithmetic for every integer trace $3 \leq t \leq 80$ — 78 traces, 0 failures (§3, [App. A]). At the first four hyperbolic traces the conjugacy classes are known exhaustively by an independent word census [App. A]:

t	conjugacy classes (cyclic words)	classes	unoriented closed geodesics
3	RL	1	1 (reciprocal)
4	R^2L, RL^2	2	1 (inverse pair)
5	R^3L, RL^3	2	1 (inverse pair)
6	R^4L, RL^4, R^2L^2	3	2 (inverse pair, and one reciprocal)

Equation (8.5) is the extreme case of the section: the ring is $\mathbb{Z}$, and the realized set is the whole of it. Nothing that follows is like it.

8.3 The realized set of $G(4)$, in closed form

$G(4)$ is presented by its exact matrix model, which splits it into two classes of elements according to where the $\sqrt{2}$ sits:

$$\text{Type I:} \quad \begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix}, \qquad a,b,c,d \in \mathbb{Z}, \quad ad - 2bc = 1; \tag{8.7}$$

$$\text{Type II:} \quad \begin{pmatrix} a\sqrt{2} & b \\ c & d\sqrt{2} \end{pmatrix}, \qquad a,b,c,d \in \mathbb{Z}, \quad 2ad - bc = 1. \tag{8.8}$$

Every element of $G(4)$ is of exactly one of the two Types, and the Types multiply by

$$\mathrm{I}\cdot\mathrm{I}=\mathrm{I},\qquad \mathrm{I}\cdot\mathrm{II}=\mathrm{II},\qquad \mathrm{II}\cdot\mathrm{II}=\mathrm{I}, \tag{8.9}$$

a $\mathbb{Z}/2$ grading of the group.

Theorem 8.2 (the realized set of $G(4)$). *Up to sign,*

$$\Theta\big(G(4)\big) \;=\; 2\mathbb{Z} \;\cup\; \sqrt{2}\cdot\mathbb{Z}, \tag{8.10}$$

and every value of the right-hand side is realized.

Proof.

Step 1 (containment). In Type I the relation $ad-2bc=1$ forces ad odd, hence a and d both odd, hence

$$\mathrm{tr} \;=\; a+d \;\in\; 2\mathbb{Z}; \tag{8.11}$$

in Type II the trace is

$$\mathrm{tr} \;=\; (a+d)\sqrt{2} \;\in\; \sqrt{2}\mathbb{Z}. \tag{8.12}$$

Since every element is of one of the two Types, $\Theta(G(4))\subseteq 2\mathbb{Z}\cup\sqrt{2}\mathbb{Z}$. In particular **no odd integer is the trace of any element of $G(4)$.**

Step 2 (realization). Three families of witnesses, each of determinant exactly 1: for the even trace $2k+2$ with $k\geq 0$, and for the trace 0,

$$\begin{pmatrix}1 & k\sqrt{2}\\ \sqrt{2} & 2k+1\end{pmatrix},\qquad \begin{pmatrix}1 & \sqrt{2}\\ -\sqrt{2} & -1\end{pmatrix}\qquad \text{(Type I)}; \tag{8.13}$$

and for the trace $m\sqrt{2}$, every $m\in\mathbb{Z}$,

$$\begin{pmatrix}0 & 1\\ -1 & m\sqrt{2}\end{pmatrix}\qquad \text{(Type II)}. \tag{8.14}$$

Step 3 (the witnesses lie in the group). Each family is of the shape (8.7) resp. (8.8) with integer a,b,c,d, hence lies in $G(4)$ by the matrix model. ∎

Verification. The consequences of (8.10) were corroborated over all 21,214 distinct PSL elements of word length ≤ 14, with zero exceptions [App. A]. The containment half was separately derived on a second enumeration of 6,268 distinct elements of word length ≤ 12, which printed the realized ladder $|t|\in\{0,\sqrt{2},2,2\sqrt{2},4,3\sqrt{2},4\sqrt{2},6,5\sqrt{2},8,6\sqrt{2},7\sqrt{2},10\}$ and found no odd integer and nothing outside $2\mathbb{Z}\cup\sqrt{2}\mathbb{Z}$ [App. A].

The two progressions of (8.10) are not two halves of an accident; they are the two cosets of $\Gamma_0(2)$ in $\Gamma_0(2)^+$, seen through one conjugation.

Proposition 8.3 (the grading is the coset grading). *Let* $g = \mathrm{diag}(1, \sqrt{2})$*, so that*

$$g\,M\,g^{-1} \;=\; \begin{pmatrix} m_{11} & m_{12}/\sqrt{2} \\ \sqrt{2}\,m_{21} & m_{22} \end{pmatrix}. \tag{8.15}$$

Then, identically in a, b, c, d*,*

$$g\begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix}g^{-1} \;=\; \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}, \tag{8.16}$$

which is integral with lower-left entry $\equiv 0 \bmod 2$ *and determinant* $ad - 2bc = 1$*, so Type I lands in* $\Gamma_0(2)$*; and*

$$g\begin{pmatrix} a\sqrt{2} & b \\ c & d\sqrt{2} \end{pmatrix}g^{-1} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 2a & b \\ 2c & 2d \end{pmatrix}, \tag{8.17}$$

of determinant $\frac{1}{2}(4ad - 2bc) = 2ad - bc = 1$*, so Type II lands in the Fricke coset* $w_2\,\Gamma_0(2)$*, where*

$$w_2 \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \tag{8.18}$$

[7]. Consequently the $\mathbb{Z}/2$ *grading (8.9) is the coset grading of* $\Gamma_0(2)^+$*: even-coset traces are rational and odd-coset traces are* $\sqrt{2}$*-proportional.*

Proof. Both (8.16) and (8.17) are the single computation (8.15) performed on (8.7) and (8.8); the determinant and the congruence on the lower-left entry are read off the results. ■

Verification. (8.16) and (8.17) are symbolic in (a, b, c, d) and carry no bound. They were additionally checked on the witness families (8.13) and (8.14) over $k, m \in [-4, 8]$: every image of the right shape, determinant 1, and trace preserved at $2k + 2$ and $m\sqrt{2}$ respectively — a bounded control on an unbounded derivation [App. A].

Remark 8.4 (the union is not disjoint, and the grading has one exception). The two progressions meet:

$$2\mathbb{Z} \cap \sqrt{2}\,\mathbb{Z} \;=\; \{0\}. \tag{8.19}$$

The union in (8.10) is therefore *not* disjoint, and the coset grading of the trace set holds **away from** $t = 0$: every non-zero trace lies in exactly one progression, while $t = 0$ lies in both and is realized in both cosets — by S in Type II and by $\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$ in Type I.

Corollary 8.5 (what (8.10) settles at once). *In* $G(4)$*:*

(i) the elliptic traces are exactly

$$\{0, \pm\sqrt{2}\}, \tag{8.20}$$

so the torsion has order 2 *or* 4 *and nothing else;*

(ii) the parabolic value $|t| = 2$ occurs in the even coset only, since $\sqrt{2}\,m = 2$ has no integer solution;

(iii) the intervals $(\sqrt{2}, 2)$ and $(2, 2\sqrt{2})$, of widths

$$2 - \sqrt{2} \;=\; 0.585786438\ldots, \qquad 2\sqrt{2} - 2 \;=\; 0.828427125\ldots, \tag{8.21}$$

contain no trace of $G(4)$ at all — the two discreteness margins flanking the parabolic value;

(iv) the minimal hyperbolic element, of trace $2\sqrt{2}$ and translation length ℓ, is primitive: a square root of it would have trace

$$2\cosh(\ell/4) \;=\; 2.197368227\ldots, \tag{8.22}$$

which lies in the interval $(2, 2\sqrt{2})$ of (iii) and is therefore not a trace of $G(4)$.

Verification. No element of the word-length-≤ 14 ball lies in either interval of (iii); the count is 0 [App. A].

Remark 8.6 (what (8.10) is not a consequence of). The closed form is not the $q = 4$ member of a family of such laws, and it is not entailed by the rationality of $\lambda_4^2 = 2$. What that rationality gives is the integrality of $\mathrm{tr}\big((TST)^2\big)$ — an associated fact at the level of a single trace. The only known route from $\lambda_4^2 \in \mathbb{Q}$ to the *full* trace set runs through commensurability, which is not the mechanism used above: (8.10) is proved from the matrix model and nothing else.

8.4 Trace gaps in the ring

Theorem 8.1 leaves no gap: at $X(1)$ the ambient ring is $\mathbb{Z}$ and every element of it is a trace. Theorem 8.2 leaves a large one, and names it exactly.

Corollary 8.7 (the gaps of $G(4)$). *In the ambient trace ring $\mathbb{Z}[\sqrt{2}]$,*

$$\mathbb{Z}[\sqrt{2}] \setminus \Theta\big(G(4)\big) \;=\; \{\, a + b\sqrt{2} \;:\; a, b \in \mathbb{Z},\ ab \neq 0 \,\} \,\cup\, (2\mathbb{Z} + 1). \tag{8.23}$$

That is: the trace gaps of $G(4)$ are exactly the mixed elements and the odd integers.

Proof. An element of $\mathbb{Z}[\sqrt{2}]$ is $a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$, and by (8.10) it is a trace precisely when $b = 0$ with a even, or $a = 0$. The complement of that condition is the right-hand side of (8.23). ■

Two features of (8.23) are worth separating. The odd integers are excluded by a group-wide parity law — the relation $ad - 2bc = 1$ of (8.7), which forces every even-coset trace to be even — so their absence has a one-line mechanism. The mixed elements are excluded by the shape of (8.10) itself, with no separate argument. Neither exclusion is claimed here for any other stage of the family: whether the absence of mixed traces is a structural feature of the even-q stages or an artefact of the bounded enumerations that first exhibited it is not settled, and §11 carries it as an open item, together with what is open about $G(5)$, whose realized set is not known in closed form.

8.5 An absent value is absent outright

A trace gap of G is not a value that G approaches and fails to reach. The distinction is sharp, and it is sharp because two layers of quantity behave differently along the trace line.

Proposition 8.8 (the two layers). *Let t range over the ambient trace interval (8.3).*

(a) The quantities belonging to the ambient group $\mathrm{PSL}(2,\mathbb{R})$ *vary continuously in t: the surd $\sqrt{|t^2-4|}$, the rotation angle* $2\arccos(t/2)$ *and the translation length* $2\operatorname{arccosh}(t/2)$*. At the seam both vanish at the same rate,*

$$\sqrt{(2-t)(2+t)} \;=\; 2\sqrt{\delta}\,\big(1+O(\delta)\big), \qquad \delta = 2-t, \tag{8.24}$$

and the ratio of the angle to the surd tends to 1*: at* $\delta = 10^{-1}, 10^{-2}, 10^{-4}, 10^{-6}$ *it is* 1.017007305*,* 1.001670007*,* 1.000016667*,* 1.000000167*.*

(b) The quantities belonging to the discrete group are supported on $\Theta(G)$ and take no intermediate values. For $G = \mathrm{PSL}(2,\mathbb{Z})$ the existence function on the elliptic trace line is the indicator of $\{0,\pm1\}$*:*

$$\mathbb{1}\big[\,\exists\, g \in G \text{ with } \operatorname{tr} g = t\,\big] \;=\; \mathbb{1}_{\{0,\pm1\}}(t), \qquad |t| < 2, \tag{8.25}$$

with no tail and no attenuation — at each of the five probed values 0.5*,* 1.5*,* $\sqrt{2}$*,* $\sqrt{3}$*,* 1.99 *there is no element, no conjugacy class, no cone point, no discriminant and no trace-formula term.*

Proof. For (a), $\sqrt{(2-t)(2+t)} = \sqrt{\delta(4-\delta)} = 2\sqrt{\delta}\sqrt{1-\delta/4}$, which is (8.24). For (b), the support statement is immediate from (8.6): a non-integer t is the trace of no element whatever, so every invariant attached to trace-t elements is attached to the empty set. ■

Verification. (8.24) was checked against the closed form at $\delta = 10^{-2}$, where the surd is 0.199749844 against $2\sqrt{\delta} = 0.200000000$; the four ratios quoted in (a) are machine values [App. A]. The enumeration in (b) is over the five probed values named there and is not a statement about a larger set.

Remark 8.9 (the approach is empty). The same shape holds from the hyperbolic side. There is no trace of $\mathrm{PSL}(2,\mathbb{Z})$ in the interval $(2,3)$: the modular surface does not degenerate toward the seam but stops, whole, at $t = 3$. The consequence is global — the surface has a length gap, with no closed geodesic shorter than 1.924847300 ..., and the trace-3 class realizes it.

8.6 Which values are traces of nothing at all

Fixing G and asking which values it misses is one question; asking which values are missed by *every* discrete group is another, and it has a complete answer on each side of the seam. Call $t \in \mathbb{R}$ **universally unrealized** if no discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ contains an element of trace t.

Theorem 8.10 (the elliptic criterion). *Let $|t| < 2$. Then t is realized in some discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *if and only if*

$$t \;=\; 2\cos(k\pi/m) \qquad \text{for some integers } k, m. \tag{8.26}$$

Moreover every such value is realized inside the Hecke family itself: $G(m)$ realizes all of $\{2\cos(k\pi/m) : 1 \le k < m\}$, as the traces of the powers U^k of its elliptic generator.

Proof.

Step 1 (necessity). An elliptic element of a discrete subgroup has finite order: an infinite-order elliptic generates an irrational rotation subgroup whose closure is all of SO(2), which is not discrete. An elliptic element of order m has trace $2\cos(k\pi/m)$, which is (8.26).

Step 2 (sufficiency, and where it is realized). For each m the Hecke group $G(m)$ of §1 is discrete and contains an elliptic element U of order m, whose powers have traces

$$\operatorname{tr} U^k \;=\; 2\cos(k\pi/m), \qquad 1 \le k < m, \tag{8.27}$$

so the whole of the set in (8.26) occurs, and occurs in this one family; at $m = q$ the right-hand side of (8.27) is the elliptic realized set (8.4) of $G(q)$. ■

Corollary 8.11 (the elliptic interval is almost entirely unrealized). *All but countably many points of $(-2, 2)$ are universally unrealized. In particular every rational t with $|t| < 2$ other than $0, \pm 1$ is universally unrealized, since the only rational values of $2\cos(r\pi)$ with $r \in \mathbb{Q}$ are $0, \pm 1, \pm 2$ [11].*

Proof. The set $\{2\cos(k\pi/m) : k, m \in \mathbb{Z}\}$ of Theorem 8.10 is a countable union of finite sets, hence countable, while $(-2, 2)$ is a continuum; the complement of a countable set in a continuum is co-countable. ■

The statement of Corollary 8.11 is exact rather than approximate, and is stated so deliberately: "all but countably many" entails measure zero without requiring a measure to be chosen, and no measure on the trace interval is used anywhere in this section.

The hyperbolic side does not merely have a larger realized set. It has no universally unrealized value at all, and the proof exhibits the group.

Theorem 8.12 (no hyperbolic trace is universally unrealized). *For every real $t > 2$ there is a discrete subgroup of* PSL$(2, \mathbb{R})$ *containing an element of trace t.*

Proof.

Step 1 (the witness). Let A be any hyperbolic element of PSL$(2, \mathbb{R})$ of trace t, and consider the cyclic group $\langle A \rangle$. It translates along the axis of A by

$$\ell \;=\; 2\operatorname{arccosh}(t/2) \;>\; 0. \tag{8.28}$$

Step 2 (the witness is discrete). The orbit of a point of the axis under $\langle A \rangle$ is

$$\{ n\ell \;:\; n \in \mathbb{Z} \}, \tag{8.29}$$

which is discrete in the axis because $\ell > 0$; hence $\langle A \rangle$ is a discrete subgroup — an elementary Fuchsian group of hyperbolic type — containing an element of trace t.

Step 3 (why the elliptic argument has no counterpart here). The obstruction in Step 1 of Theorem 8.10 was that an infinite-order elliptic has orbit dense in SO(2). There is no analogue: (8.29) is discrete for *every* $\ell > 0$. ■

The mechanism is one line: **discreteness constrains rotation angles and does not constrain translation lengths.**

> **Remark 8.13 (the answer under three reference classes).** Theorem 8.12 quantifies over all discrete subgroups, and the reader may object that this is too generous a class. It is not the only class in which the question is well posed, and the answer does not change:
>
> | reference class | hyperbolic side | elliptic side |
> |---|---|---|
> | any discrete subgroup | every $t > 2$ realized | $t = 0.5$ realized in none |
> | any lattice (finite covolume) | every $t > 2$ realized | unchanged |
> | the Hecke family only | only algebraic t — a thin subset | only $2\cos(k\pi/q)$ |
>
> Under the third reading everything is thin, the elliptic side included, so that reading grades the *family* and not the unrealized set. The asymmetry survives every reference class in which the question is well posed.

The section therefore closes on a dichotomy that can be stated in one sentence and is proved in two: **the realized subset of the elliptic ambient interval is countable, with co-countable complement (Theorem 8.10, Corollary 8.11); the realized subset of the hyperbolic ambient interval is the whole of it (Theorem 8.12).** Both are statements about the ambient interval (8.3) and neither is a statement about the ambient trace ring (8.2), which is countable on both sides — the distinction §8.1 was set up to keep.

9. The value identity and the mutual-exclusion theorem

At one of the three places where the two families share a field, the sharing is an equality of numbers and not merely of fields: the parameter of the Hecke group $G(5)$ *is* the fundamental unit of $\mathbb{Q}(\sqrt{5})$. This section proves that equality, names the classical mechanism that produces it, and then proves that no group relation can accompany it — at any Hecke group whose trace field is quadratic, "the parameter is a unit" and "the group is arithmetic" exclude one another. The Hecke groups with quadratic trace field are exactly those with $q \in \{4, 5, 6\}$ (Theorem 2.10), so the two other than $G(5)$ are not a sample but the rest of the population, and their outcomes are entailed rather than observed. What the exclusion leaves standing is the equality itself, which is exact; §9.3 states the difference between the two, because that difference is this section's point.

9.1 The comparison, and the type it must have

As in §1, write

$$\lambda_q \;=\; 2\cos(\pi/q), \qquad q \geq 3, \tag{9.1}$$

for the parameter of the Hecke group $G(q) = H(\lambda_q)$, and $\mathbb{Q}(\lambda_q)$ for its trace field. At $q = 5$ the parameter is the golden ratio,

$$\lambda_5 \;=\; 2\cos(\pi/5) \;=\; \tfrac{1}{2}\big(1+\sqrt{5}\big) \;=\; \varphi, \tag{9.2}$$

and φ occurs a second time in the neighbourhood, as an endpoint of the axis of the trace-3 class on $X(1)$ — the class that realizes the systole (§5). The comparison, as it is naturally posed, sets these two occurrences side by side.

They are not of the same type. The first is a trace: a class function on $G(5)$, invariant under conjugation, valued in the trace field. The second is a point of $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, a set on which the group acts, and it moves: a conjugate γ carries the endpoint to $\gamma(\varphi)$. An axis endpoint is therefore not well defined until a representative matrix is chosen. With

$$T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad T^{-1}AT \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \tag{9.3}$$

the two matrices of (9.3) lie in one conjugacy class of $\mathrm{PSL}(2,\mathbb{Z})$ and both have trace 3, while their fixed-point pairs are

$$\big\{\varphi,\ -\varphi^{-1}\big\} \qquad \text{and} \qquad \big\{\tfrac{1}{2}(-1+\sqrt{5}),\ \tfrac{1}{2}(-1-\sqrt{5})\big\} \tag{9.4}$$

respectively. So "the endpoint is φ" is a statement about a chosen matrix, not about the class, and a conjugacy invariant and a non-invariant cannot be one object. The comparison has to be remade invariant-to-invariant before it can be decided at all, and the invariant form of it is

$$\lambda_5 \;=\; \varepsilon_5, \tag{9.5}$$

the parameter of $G(5)$ against the fundamental unit of $\mathbb{Q}(\sqrt{5})$ — two invariants, and a decidable statement. (Fundamental units are indexed by field discriminant throughout, as in §3; $\mathbb{Q}(\sqrt{5})$ has discriminant 5.)

9.2 The identity, and why it is not a coincidence

Theorem 9.1. *The identity (9.5) holds: the parameter of $G(5)$ is the fundamental unit of $\mathbb{Q}(\sqrt{5})$, and*

$$N(\varepsilon_5) \;=\; -1. \tag{9.6}$$

Proof.

Step 1 (conductor–discriminant). $\mathbb{Q}(\sqrt{5})$ has discriminant 5, hence conductor 5, hence

$$\mathbb{Q}(\sqrt{5}) \;\subset\; \mathbb{Q}(\zeta_5). \tag{9.7}$$

Since $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$ and $\mathbb{Q}(\sqrt{5})$ is real of degree 2, it is the maximal real subfield:

$$\mathbb{Q}(\sqrt{5}) \;=\; \mathbb{Q}(\zeta_5)^+. \tag{9.8}$$

Step 2 (λ_5 is a Gaussian period).

$$2\cos(\pi/5) \;=\; -\big(\zeta_5^2 + \zeta_5^3\big), \tag{9.9}$$

a real cyclotomic number at level 5: an element of $\mathbb{Q}(\zeta_5)^+$ manufactured by the roots of unity alone.

Step 3 (it is a unit, and it is *the* unit). With Φ_q the q-th cyclotomic polynomial,

$$\big|N(\lambda_5)\big| \;=\; \big|\Phi_5(-1)\big| \;=\; 1, \tag{9.10}$$

and λ_5 satisfies

$$x^2 - x - 1, \tag{9.11}$$

whose constant term gives $N(\lambda_5) = -1$. Since $\mathbb{Q}(\sqrt{5})$ has class number 1, its cyclotomic unit is the fundamental unit up to sign and powers; hence $\lambda_5 = \varepsilon_5$. ■

Verification. The Gaussian-period identity (9.9) was checked in closed form, and the identification (9.5) was then re-checked by a route that does not assume it: a fundamental-unit routine called at discriminant 5 returns $(u, v, \mathrm{den}, N) = (1, 1, 2, -1)$ — that is, $\frac{1}{2}(1+\sqrt{5})$ with norm -1 — produced independently of the symbol φ. The instrument reports 13 independent closed-form checks, 13 true, 0 false; two further comparisons in the original tally compared a value with itself and are excluded from that count [App. A].

> **Remark 9.2 (what Theorem 9.1 adds to (9.2)).** The equality $2\cos(\pi/5) = \frac{1}{2}(1+\sqrt{5})$ in (9.2) is exact, but on its own it is a coincidence of values. Steps 1–3 say that it is not one: the fundamental unit of $\mathbb{Q}(\sqrt{5})$ is a cyclotomic unit at level 5, and λ_5 is that cyclotomic unit. The mechanism is classical, and it is the mechanism — not the numerical agreement — that Theorem 9.1 asserts.

The same field is reached along the two mechanism families of §2, and they agree for one identifiable reason.

Proposition 9.3 (the two routes to $\sqrt{5}$). *The torsion half-product at order* 5 *and the traveling surd at trace* 3 *(§2.1) return*

$$\varphi + \varphi^{-1} \;=\; \sqrt{5} \qquad \text{and} \qquad \varphi^2 - \varphi^{-2} \;=\; \sqrt{5} \tag{9.12}$$

respectively. Since

$$\varphi^2 - \varphi^{-2} \;=\; (\varphi - \varphi^{-1})(\varphi + \varphi^{-1}), \tag{9.13}$$

the two expressions in (9.12) are equal if and only if

$$\varphi - \varphi^{-1} \;=\; 1, \qquad \text{i.e.} \qquad \varphi^2 - \varphi - 1 \;=\; 0. \tag{9.14}$$

The two mechanisms therefore return the same number for exactly one reason: the minimal polynomial of φ.

Remark 9.4 (one fact, stated twice). For a monic quadratic $x^2 - sx + n$ the norm of a root is the constant term n. So "the minimal polynomial (9.11) of φ has constant term -1" — the reason the two mechanisms of Proposition 9.3 agree — and "λ_5 is a unit of norm -1" — Step 3 of Theorem 9.1 — are one statement in two forms. The seam between the torsion and traveling routes to $\sqrt{5}$ is invisible because φ's minimal polynomial closes it.

9.3 The mutual-exclusion theorem

Theorem 9.5 (mutual exclusion). *Let $q \geq 3$ be such that the trace field $\mathbb{Q}(\lambda_q)$ is quadratic. Then the two statements*

(i) λ_q is a unit;

(ii) $G(q)$ is arithmetic

cannot both hold.

Proof, in three steps.

Step 1 (arithmeticity forces $\lambda_q^2 \in \mathbb{Q}$). $G(q)$ contains the parabolic $z \mapsto z + \lambda_q$ and is therefore non-cocompact; a non-cocompact arithmetic Fuchsian group is commensurable with $\mathrm{PSL}(2,\mathbb{Z}) = G(3)$, and the invariant trace field is a commensurability invariant [17, ch. 4]. Since

$$k\Gamma\big(G(q)\big) \;=\; \mathbb{Q}(\lambda_q^2), \qquad k\Gamma\big(G(3)\big) \;=\; \mathbb{Q}, \tag{9.15}$$

hypothesis (ii) gives $\lambda_q^2 \in \mathbb{Q}$.

Step 2 ($\lambda_q^2 \in \mathbb{Q}$ forces a bare square root). For λ_q of degree 2,

$$\lambda_q^2 \in \mathbb{Q} \iff \operatorname{tr}\lambda_q = 0 \iff \lambda_q = \sqrt{m} \quad \text{for an integer } m \geq 2 \text{ that is not a square.} \tag{9.16}$$

Step 3 (a bare square root is not a unit).

$$N(\sqrt{m}\,) \;=\; -m, \tag{9.17}$$

so hypothesis (i) would force $m = 1$, contradicting $m \geq 2$ in (9.16). Hence (i) and (ii) cannot both hold. ■

The theorem's content is narrow, and the narrowness is the point.

Remark 9.6 (what Theorem 9.5 does not say). Three things.

(a) It does not touch the identity of Theorem 9.1. The equality $\lambda_5 = \varepsilon_5$ is exact, and it has the mechanism of §9.2. What Theorem 9.5 excludes is a *relation between the two groups*, not the *equality of the two numbers*; the numbers coincide in $\mathbb{Q}(\sqrt{5})$ and the groups remain unrelated. Nothing proved here weakens (9.5), and nothing in (9.5) weakens what is proved here.

(b) It is one empty cell of a $2{\times}2$, not a correlation. The forbidden combination is (unit *and* arithmetic). The theorem asserts nothing about the combination (non-unit *and* non-arithmetic). Among $q = 4, 5, 6$ the converse implication "non-unit ⇒ arithmetic" does hold, but it holds there by inspection and with no mechanism, and it is false for Hecke groups generally: at $q = 8$ and $q = 10$ the parameter is not a unit and the group is not arithmetic. The result is accordingly a **mutual exclusion** — one forbidden combination — and is called that throughout; it is not a correlation between the two properties, and no name suggesting one is used for it.

(c) The hypothesis is load-bearing. Step 2 uses $\deg\lambda_q = 2$. Nothing above constrains a Hecke group whose trace field has degree ≥ 3.

Which parameters are units is settled by a rule that mentions no parity.

Proposition 9.7 (the unit rule). *λ_q is a unit if and only if $|\Phi_q(-1)| = 1$, if and only if q is **not** of the form $2p^k$ with p prime and $k \geq 1$; indeed*

$$|\Phi_q(-1)| \;=\; \begin{cases} p, & q = 2p^k, \\ 1, & \text{otherwise.} \end{cases} \tag{9.18}$$

In particular, among the three groups of quadratic trace field,

$$4 = 2\cdot 2^1, \qquad 6 = 2\cdot 3^1, \qquad 5 \neq 2p^k, \tag{9.19}$$

so $q = 5$ carries the unique unit.

Verification. The rule was recomputed over $q = 3 \dots 30$ — exhaustive in that range, 0 failures — and the consequence (9.19) checked separately [App. A]. Over the same range the even units are $q = 12, 20, 24, 28, 30$ and the non-units are $q = 4, 6, 8, 10, 14, 16, 18, 22, 26$, every one of them of the form $2p^k$. A parity form of the rule would be false: $q = 12$ is an even unit.

9.4 The complete enumeration

The two Hecke groups other than $G(5)$ are not a control group. By the organizing theorem of §2 (Theorem 2.10),

$$\left\{ q \geq 3 \ : \ \deg_{\mathbb{Q}} \mathbb{Q}(\lambda_q) = 2 \right\} \ = \ \{4,\ 5,\ 6\}, \tag{9.20}$$

verified over $q = 3 \dots 200$ with 0 failures [App. A]. So $\{4, 6\}$ is the two-element complement of the test case inside a three-element universe: running it completes an enumeration rather than tests a hypothesis, and both outcomes are entailed by Theorem 9.5 before they are computed. They are reported because an enumeration should be exhibited.

	$G(4)$	$G(5)$	$G(6)$
λ_q	$\sqrt{2}$	φ	$\sqrt{3}$
minimal polynomial	$x^2 - 2$	$x^2 - x - 1$	$x^2 - 3$
trace field	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{3})$
field discriminant	8	5	12
$N(\lambda_q) = \pm\Phi_q(-1)$	-2	-1	-3
λ_q a unit?	no	**yes**	no
$\lambda_q = \varepsilon$?	no ($\varepsilon_8 = 1 + \sqrt{2}$)	**yes**	no ($\varepsilon_{12} = 2 + \sqrt{3}$)
$\lambda_q^2 \in \mathbb{Q}$?	yes ($= 2$)	no ($= \varphi + 1$)	yes ($= 3$)
$k\Gamma = \mathbb{Q}(\lambda_q^2)$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}$
$G(q)$ arithmetic?	yes	**no**	yes
commensurable with $\mathrm{PSL}(2, \mathbb{Z})$?	yes	**no**	yes

Read down the last four rows: the combination Theorem 9.5 forbids is empty, and it is empty across the whole population rather than across a sample of it. The arithmeticity entries agree with the classification of the arithmetic Hecke groups [1], which §4 carries together with the routes to it.

9.5 What the identity transports

Proposition 9.8 (the bound). *The invariant trace field is a commensurability invariant [17, ch. 4], and it separates the two groups at issue:*

$$k\Gamma\big(G(3)\big) = \mathbb{Q}, \qquad k\Gamma\big(G(5)\big) = \mathbb{Q}\big(\lambda_5^2\big) = \mathbb{Q}(\varphi + 1) = \mathbb{Q}(\sqrt{5}). \tag{9.21}$$

Two groups with different invariant trace fields share no finite-index subgroup. Therefore no length, area, spectrum or cone datum transports across the identity (9.5).

One derived invariant is the whole warrant, and it has to be: the differences one would reach for first carry no information.

Remark 9.9 (three differences that warrant nothing). Let $\Gamma_0(2) < \mathrm{PSL}(2,\mathbb{Z})$, a subgroup of index 3 and hence commensurable with $\mathrm{PSL}(2,\mathbb{Z})$ by construction. Then

	$X(1)$	$\Gamma_0(2)$	
area	$\pi/3$	$3 \cdot \pi/3 = \pi$	differ
cone orders	$\{2,3\}$	$\{2\}$	differ
systole	$4\log\varphi$ (§5)	a different value	differ

Three properties that differ across a commensurable pair support nothing about non-commensurability. Proposition 9.8 uses none of them.

Remark 9.10 (the bridging object is φ, not $\sqrt{5}$). By (9.17) at $m = 5$,

$$N(\sqrt{5}) \;=\; -5, \tag{9.22}$$

so $\sqrt{5}$ is not a unit. The number that equals the fundamental unit in (9.5) is φ; the surd $\sqrt{5}$, which is what the two mechanisms of Proposition 9.3 both return, is precisely the object the identity does not carry.

The verdict has a shape worth stating plainly. The two occurrences of §9.1 are not one object reached by two mechanisms: they are of different types, they live on groups sharing no finite-index subgroup by Proposition 9.8, and the mechanisms that produce them share no step. Nor are they two objects that happen to share a value, if *happen* is to mean coincidence: the shared value has the classical mechanism of Theorem 9.1, and Proposition 9.3 exhibits the single identity that closes the seam between the two routes. What remains is the third description, and it is the accurate one — two objects, on unrelated groups, joined by a theorem that lives entirely inside $\mathbb{Q}(\sqrt{5})$ and touches neither group. The theorem would hold if neither $\mathrm{PSL}(2,\mathbb{Z})$ nor $G(5)$ had ever been defined.

10. The parabolic trace and the primitivity filter

One trace value of the modular group behaves unlike the rest. At $t = 2$, where the elements are parabolic, the conjugacy classes are indexed by a nonzero integer, so a single trace carries infinitely many of them; at every hyperbolic trace the count is finite. This section shows that the difference is an artifact of counting proper powers. The classes at $t = 2$ are the powers of two primitive ones; under the primitivity filter the parabolic count is 2, every trace value of $X(1)$ carries finitely many primitive classes, and what remains at $t = 2$ is an infinity of repetition rather than of distinct structure.

10.1 The classes at $t = 2$

Let T be the translation matrix of (3.5).

> **Proposition 10.1 (the parabolic classes).** *The conjugacy classes of parabolic elements of* $\mathrm{PSL}(2,\mathbb{Z})$ *are represented bijectively by T^n, $n \in \mathbb{Z}\setminus\{0\}$. In particular the single trace value $t = 2$ carries infinitely many classes, and T^n is conjugate to T^m if and only if $n = m$; T is not conjugate to T^{-1}.*

Proof.

Step 1 (the conjugation formula). For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, direct multiplication with $g^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and $ad - bc = 1$ gives

$$g\,T^n g^{-1} \;=\; \begin{pmatrix} 1 - acn & a^2 n \\ -c^2 n & 1 + acn \end{pmatrix}. \tag{10.1}$$

Step 2 ($T^n \sim T^m$ forces $n = m$). If $g\,T^n g^{-1} = T^m$, the lower-left entry of (10.1) gives $c^2 n = 0$, hence $c = 0$ since $n \neq 0$; then $ad = 1$ with $a, d \in \mathbb{Z}$ gives $a = d = \pm 1$, so $a^2 = 1$, and the upper-right entry of (10.1) gives

$$m \;=\; a^2 n \;=\; n. \tag{10.2}$$

Conjugacy to $-T^m$ is impossible in $\mathrm{SL}(2,\mathbb{Z})$, the trace being a conjugation invariant and $2 \neq -2$; in $\mathrm{PSL}(2,\mathbb{Z})$ one has $[T^n] \sim [T^m]$ exactly when $g\,T^n g^{-1} = \pm T^m$ for some g, and both branches force $n = m$.

Step 3 (every parabolic is conjugate to some T^n). A parabolic P fixes a unique point $x \in \mathbb{Q} \cup \{\infty\}$; by the transitivity of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{Q} \cup \{\infty\}$ choose γ with $\gamma x = \infty$. Then $\gamma P \gamma^{-1}$ fixes ∞, hence is upper triangular with determinant 1, that is $\pm T^n$ with $n \neq 0$. ■

Verification. The identity (10.1) was checked in exact integer arithmetic for all 116 determinant-one integer matrices with entries of absolute value at most 3 and $n \in \{1, 2, -1, 5\}$ [App. A]. Independently, an exhaustive search over all $g \in \mathrm{SL}(2,\mathbb{Z})$ with entries of absolute value at most 8 and $n, m \in [-3, 3] \setminus \{0\}$, testing $g\,T^n g^{-1} = \pm T^m$, returned exactly the diagonal pairs (n, n) with the $+$ sign and no cross pair — no $(1, -1)$, no $(1, 2)$ [App. A]. The search bounds the conjugators only; Step 2 closes the statement for all of them. That the class representatives genuinely leave the upper-triangular family under conjugation is witnessed by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{10.3}$$

a parabolic fixing 0 and of no form $\pm T^m$, which returns under further conjugation only to the same T^n.

Remark 10.2 (the stratification is arithmetic). The integer index of Proposition 10.1 is a feature of the discrete group, not of the ambient one. In $\mathrm{PSL}(2, \mathbb{R})$ there are exactly two parabolic classes, since conjugation by $\mathrm{diag}(\lambda, \lambda^{-1})$ rescales n by $\lambda^2 > 0$ and so collapses all $n > 0$ together and all $n < 0$ together. In $\mathrm{GL}(2, \mathbb{Z})$ the further conjugation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T^{-1} \tag{10.4}$$

merges n with $-n$ — but that conjugator has determinant -1 and is not available inside $\mathrm{PSL}(2, \mathbb{Z})$, which is why the sign survives as an invariant in Proposition 10.1.

10.2 Primitivity, and what remains

Call a group element *primitive* when it is not a proper power of another element.

Corollary 10.3 (two primitive classes at $t = 2$). *Exactly two of the classes of Proposition 10.1 are primitive, namely $n = +1$ and $n = -1$; every other class is a power of one of these.*

This is immediate from the rigidity of Proposition 10.1: T^n is the n-th power of T, and no two of the T^n are conjugate.

Remark 10.4 (two senses of primitive). Two notions of primitivity occur in this paper and they are not the same. A *form* is primitive when its content — the gcd of its coefficients, as in §3.1 — is 1; an *element* is primitive in the sense just defined. The counts differ: over $t = 3, \dots, 12$ the number $h^+(t^2 - 4)$ of classes of primitive forms of discriminant $t^2 - 4$ and the number of element-primitive classes disagree at $t = 6, 10, 11$. Throughout this section *primitive* means element-primitive.

Theorem 10.5 (finiteness under the primitivity filter). *On $X(1)$, every trace value carries finitely many primitive classes — the parabolic value $t = 2$ included, where the count is 2 by Corollary 10.3. At every hyperbolic trace the primitive count is finite because the total class count is finite — the number of classes of binary quadratic forms of a given discriminant being finite, which is classical [5, 6] and is not derived here — and the primitive count is bounded above by it.*

Verified by explicit count at $t = 3, \dots, 12$. No primitivity count has been computed at $G(4)$, $G(5)$ or $G(6)$, and nothing is claimed about their trace values.

The counts were obtained from a criterion rather than asserted. Write $p_k(s)$ for the trace of the k-th power of an element of trace s, given by the power-trace recursion

$$x_0 = 2, \qquad x_1 = s, \qquad x_k \;=\; s\,x_{k-1} - x_{k-2} \quad (k \geq 2), \tag{10.5}$$

with $p_k(s) = x_k$. Then a *hyperbolic* trace t carries a non-primitive class if and only if

$$t \;=\; p_k(s) \qquad \text{for integers } s \geq 3,\ k \geq 2. \tag{10.6}$$

At $t = 2$ the non-primitive classes are the T^n with $|n| \geq 2$, by Corollary 10.3.

Over $t = 3, \dots, 12$ this gives the three counts — classes of primitive forms of discriminant $t^2 - 4$, all classes counted with orientation, and primitive classes:

t	3	4	5	6	7	8	9	10	11	12
$h^+(t^2-4)$	1	2	2	2	2	4	2	4	2	4
all classes	1	2	2	3	3	4	2	6	3	4
primitive	1	2	2	3	2	4	2	6	3	4

Verification. The element-primitive counts $1, 2, 2, 3, 2, 4, 2, 6, 3, 4$ at $t = 3, \dots, 12$ were computed twice by structurally unrelated routines — the power-trace recursion (10.5) and an independent enumeration of reduced-form cycles — and differ from the total class count at exactly one trace in that range [App. A].

> **Remark 10.6 (the one non-primitive hyperbolic trace in the range).** Over $3 \leq t \leq 12$ exactly one trace carries a non-primitive class, namely $t = 7$, the square of $X(1)$'s systolic element:
>
> $$\operatorname{tr}(A^2) \;=\; 3^2 - 2 \;=\; 7, \qquad \operatorname{tr} A = 3. \tag{10.7}$$
>
> Its discriminant factors as
>
> $$t^2 - 4 \;=\; 45 \;=\; 3^2 \cdot 5, \qquad D_0 = 5, \quad f = 3, \tag{10.8}$$
>
> so the one imprimitive trace in the range returns to $\mathbb{Q}(\sqrt{5})$ one conductor up. The identification was reached here from the recursion (10.5), and independently from a word-length census.

The trace line is infinite; every trace value on it carries finitely many primitive classes. The infinity at $t = 2$ is the infinity of $\mathbb{Z}$ acting on one object by powers — an infinity of repetition, not of distinct structure.

11. Scope, and the edges of the domain

The objects of this paper are of four kinds: the modular group; the Hecke groups $G(q)$; the Fricke extensions $\Gamma_0(p)^+$; and, attached to each of these, the conjugacy classes of its hyperbolic elements together with the traces of those classes, the fields the traces generate, and the lengths and distances the hyperbolic metric attaches to them. Every statement of §§2–10 is a statement about one of these objects, or about a map between two of them, and each is proved at the scope at which it holds.

That scope is of two kinds and is written into the statements themselves. Some results are derivations, and their scope is their hypotheses — among them Theorem 3.6, Theorem 5.5, Theorem 8.2, Theorem 9.5 and Proposition 10.1 quantify over everything their hypotheses admit, and the enumerations printed beside them are controls on the derivation rather than its evidence. The others are statements about a population — a range of traces, a range of discriminants, a syllable depth, a box of matrix entries — and there the population is the boundary of the claim and appears inside the sentence that makes it. A result of the second kind is not a weaker version of one of the first kind; it is a different statement, and the two are not interchangeable.

The domain so described has a boundary, and this section draws it. What follows is not a list of defects in the results above. Each item is a question about the same neighborhood, raised by the work rather than left over from it, and several of them are the asymptotic form of something proved here in a finite range.

11.1 The realized trace sets

The integer traces of the golden Hecke group. Remark 5.7 records the realized integer traces of $G(5)$ as $\{0, \pm 2\}$ over all cyclically reduced words of syllable length at most 12, exhaustively over conjugacy classes at that depth, and Corollary 5.6 removes 3 — with everything else in the interval $(2, 2\varphi)$ — by a derivation. Whether $\{0, \pm 2\}$ is the whole of $\Theta(G(5)) \cap \mathbb{Z}$ is open. What is settled is the kind of argument that cannot decide it.

> **Proposition 11.1 (no uniform separation).** *The non-integer realized traces of $G(5)$ come arbitrarily close to the integers:*
>
> $$\inf\big\{\, |t-k| \;:\; t \in \Theta(G(5)) \setminus \mathbb{Z},\ k \in \mathbb{Z} \,\big\} \;=\; 0. \tag{11.1}$$

Proof. Let S and T generate $G(5)$ as in §5, T the translation by φ. For all integers m, n,

$$T^m S \;=\; \begin{pmatrix} m\varphi & -1 \\ 1 & 0 \end{pmatrix}, \qquad T^m S\, T^n S \;=\; \begin{pmatrix} mn\varphi^2 - 1 & -m\varphi \\ n\varphi & -1 \end{pmatrix}, \tag{11.2}$$

so that, writing $N = mn$ and using $\varphi^2 = \varphi + 1$,

$$\operatorname{tr}\big(T^m S\, T^n S\big) \;=\; mn\varphi^2 - 2 \;=\; N\varphi + (N-2). \tag{11.3}$$

The coefficient of φ in (11.3) therefore takes every integer value. Take $N = F_j$, the j-th Fibonacci number; from Binet's formula, $F_j\varphi - F_{j+1} = -(-\varphi^{-1})^j$, so the distance from the trace (11.3) to the nearest integer is φ^{-j}, which tends to 0. ∎

No separation argument, no gap argument and no interval-arithmetic argument can therefore settle the question, however deep the enumeration: the evidence such an argument would collect is exactly what a false statement with a shrinking gap would also produce (§A.3). The computed record bears the mechanism out — at syllable depth 12 the realized non-integer trace nearest to $\mathbb{Z}$ is $4179 + 6765\varphi = 15124.999934\ldots$, whose φ-coefficient is F_{20}, and the near misses generally are the convergents of φ [App. A].

What a proof would have to be. By (11.3) the question is one about the coefficient b of φ in $\mathrm{tr}(w) \in \mathbb{Z}[\varphi]$ as a function of the word: that $b \neq 0$ for every cyclically reduced w other than the three of Remark 5.7. One necessary condition is available. Since 2 is inert in $\mathbb{Q}(\sqrt{5})$ one has $\mathbb{Z}[\varphi]/(2) \cong \mathbb{F}_4$ and $\mathrm{PSL}(2, \mathbb{F}_4) \cong A_5$; in characteristic 2 the trace fixes the order stratum, elements of order 1 or 2 having trace 0, of order 3 trace 1, and of order 5 trace ω or ω^2, which do not lie in $\mathbb{F}_2$. Hence

$$\mathrm{tr}(w) \in \mathbb{Z} \implies \text{the image of } w \text{ in } A_5 \text{ has order } 1,\ 2 \text{ or } 3,\ \text{never } 5. \tag{11.4}$$

The condition is necessary and not sufficient, and its ceiling is measured rather than estimated: of the 182,471 distinct non-integer trace values at syllable depth 12, exactly 68,758 arise from classes that also satisfy (11.4) [App. A]. Adjoining the analogous congruences at the further inert primes 3, 7 and 13 narrows the survivors without emptying them — 32 classes of syllable length at most 8 pass all four, each of them removed instead by exact arithmetic in $\mathbb{Z}[\varphi]$ [App. A]. A proof, if there is one, is an exact statement about b — a positivity law, a recursion with a sign invariant, or a congruence to a modulus growing with the word — of which (11.4) is a first instance whose reach is now known.

A closed form for $\Theta(G(5))$. Theorems 8.1 and 8.2 give the realized sets of $X(1)$ and $G(4)$ in closed form, the second graded by the two cosets of $\Gamma_0(2)$ in $\Gamma_0(2)^+$ (Proposition 8.3). Both descriptions are available because the group is a Fricke floor, and by Theorem 4.5 the Fricke floors run out at $p = 3$; $\sqrt{5} > 2$ places $G(5)$ outside that family, so its trace set is not a union of arithmetic progressions and no coset decomposition grades it. No closed form for $\Theta(G(5))$, and no structural description of it of another kind, is known.

Pure and mixed traces at the even stages. In $G(4)$ every realized trace is a rational integer or an integer multiple of $\sqrt{2}$, never a mixed value $a + b\sqrt{2}$ with $ab \neq 0$; this is proved by Theorem 8.2, the odd integers excluded by the relation $ad - 2bc = 1$ and the mixed values by the shape of (8.10) itself. At $G(5)$ mixed traces occur, $3 + \varphi$ and $4 + 3\varphi$ appearing already in the census of Proposition 7.10. Whether the absence of mixed traces is a structural feature of the even-q stages generally, or an artefact of the bounded enumerations that exhibited it beyond $q = 4$, is not settled here; neither exclusion of Theorem 8.2 is claimed for any other stage.

How large the realized sets are. The sets of §8 lie inside the trace rings $\mathbb{Z}$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\varphi]$ and $\mathbb{Z}[\sqrt{3}]$, and one may ask what proportion of a ring a group realizes. A finite enumeration — matched by population rather than by word length, at syllable depths 11, 9, 8 for $q = 4, 5, 6$, giving 265,719, 349,524 and 488,280 words, and counted in the Minkowski box $|t| \leq X$, $|t^\sigma| \leq X$ for $25 \leq X \leq 800$ — gives the following, against the ring's own exponent 2 and with each group's structural hull as the second ambient.

Table 11.1. *Two ambients, two orderings.*

q	trace ring	fitted exponent in the ring	structural hull	density in the hull
4	$\mathbb{Z}[\sqrt{2}]$	1.014	$\mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$	0.31
5	$\mathbb{Z}[\varphi]$	1.117	$\mathbb{Z}[\varphi]$	0.0005
6	$\mathbb{Z}[\sqrt{3}]$	1.023	$\mathbb{Z} \cup \sqrt{3}\,\mathbb{Z}$	0.35

The hull at $q = 5$ is the whole ring, there being no coset decomposition to confine the trace set. In the ring all three sets are sparse, their exponents clustering near 1 against the ring's 2, and the exponent of the non-arithmetic group is not the smallest of the three; in each group's own hull the picture separates the other way, $G(4)$ and $G(6)$ filling theirs while $G(5)$ occupies a proportion smaller by a factor of about 700. A comparison of sparsity across these groups is therefore a statement about the ambient as much as about the group. The asymptotics are open: the enumerations are finite, and each count carries a saturation figure recording the fraction of the enumerated set already inside the box, beyond which the exponent measures the truncation and not the group [App. A].

11.2 Levels

Completeness of the level list. Proposition 6.10 decides, for squarefree p and a rung n, whether that rung is the shortest closed geodesic of $\Gamma_0(p)^+$, and the decision is finite: only the traces t with $3 \le t \le \lfloor n_0(p)\sqrt{p} \rfloor$ need be tested. Run over every squarefree p with $2 \le p \le 20{,}000$, the criterion returns exactly the 25 levels of (6.26), the largest $p = 2046$. The mechanism of the cut-off is that $\sqrt{p}$ grows while $t_{\min}(p)$ of (6.23) does not — by Lemma 5.2 an integer trace t is realized in $\Gamma_0(p)$, p an odd prime, exactly when $t^2 - 4$ is a square modulo p, and each small t is a quadratic residue for a positive proportion of p — so beyond some point the integer progression always wins. Whether the list is complete over all p is open, and what a proof needs is an effective upper bound for $t_{\min}(p)$, that is, an effective bound on the least $t \ge 3$ with $t^2 - 4$ a quadratic residue modulo p. Deciding a single p needs no such bound; the finite criterion and the asymptotic question are independent, and the first does not wait on the second.

The multiplicity of admissible levels. Section 3.4 exhibits the trace equation $Qn^2 = t + 2$ with more than one solution — one solution at $t = 3$ and two at $t = 6$, over $Q \le 500$ and $n \le 59$ — and Proposition 3.11 accounts for the instance at $t = 6$: the two levels lie in a common normalizer, and the two solutions are two factorizations of a single coset trace rather than two geodesics. Whether the multiplicity recurs at other traces, and whether the number of admissible pairs (Q, n) at a trace obeys a counting law, is not addressed here; the enumeration behind Proposition 3.9 was carried out at two traces.

11.3 Reciprocity and threading

Theorems 7.1 and 7.4 settle the equivalence and the existence question on $X(1)$, and Proposition 7.12 settles negatively whether the unit-level partner of reciprocity survives to $G(5)$ — it fails in both directions, each direction by an exhibited witness. What those results leave open is the mechanism of that failure and the exact reach of the statement that replaces it.

The conductor. On $X(1)$ the norm of the fundamental unit implies threading and is not implied by it (Proposition 7.8); at $G(5)$ the corresponding order-level biconditional fails in both directions (Proposition 7.12). The relative discriminant of each of the two classes that break it carries a square factor,

$$12 + 16\varphi \;=\; 2^2\,(3 + 4\varphi), \qquad 6070 + 9815\varphi \;=\; (7\varphi - 1)^2\,(57 + 92\varphi), \tag{11.5}$$

so at each the offending prime divides the conductor of the order $\mathcal{O}_K[\varepsilon]$ — its index in the maximal order of L — and not the relative discriminant of L/K. Whether -1 is the norm of a unit of that particular order is sensitive to the conductor; whether -1 is a norm from the field is not. Whether the biconditional is restored on the classes whose relative discriminant is fundamental is open, and an affirmative answer would also account for the co-tracking on $X(1)$ and for the instance at $G(4)$.

Reciprocity one step outside the group. Proposition 7.10 finds the shortest closed geodesic of $G(5)$ non-reciprocal, and Remark 7.11 gives the mechanism: the pivots of a reciprocal class are half-turns, and the order-5 cone point of $G(5)$ carries none. The reversal exists nonetheless, in an orientation-reversing extension. Let $R : z \mapsto -\bar{z}$, acting on matrices by ι; then

$$\iota\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \qquad \iota(S) = S, \quad \iota(T) = T^{-1}, \quad \iota(W_0) = W_0^{-1}, \tag{11.6}$$

the last exactly, for the minimal element W_0 of (5.24). So R normalizes $G(5)$, and the class is reciprocal in $G(5) \rtimes \langle R \rangle$. The inverting element is a line reflection and not a half-turn — its fixed set is the imaginary axis, a geodesic, which meets the class's axis perpendicularly at $i\sqrt{\varphi}$ — and that is exactly why it does not lie in $G(5)$. Which groups of the neighborhood carry reciprocity inside the group and which only in the reflection extension, and what arithmetic invariant separates the two cases, is not taken up here.

What the archimedean implication characterizes. Theorem 7.13 gives, for a hyperbolic $W \in G(5)$ of trace t, that reciprocity in $G(5)$ implies $|t^\sigma| > 2$ — equivalently $\sigma(t^2 - 4) > 0$, equivalently L totally real — with 42 reciprocal classes among the 3,346 hyperbolic classes of syllable length at most 7 and no violation. Its converse is false, the class of trace 4φ having $|t^\sigma| = 2.472\ldots$ and no inverting conjugator in a ball of 152,914 elements. What the surviving implication characterizes exactly is open: whether the classes with $|t^\sigma| > 2$ that fail to be reciprocal form a structured set, and what would cut them out, is settled by neither computation behind Theorem 7.13.

The distribution of the threading traces. Proposition 7.6 classifies the 33 threading traces in $3 \le t \le 200$: fourteen with $t - 2$ a square, four with $t + 2$ a square, and fifteen in neither family. In the shorter window $3 \le t \le 40$ the two families account for eight of the ten threading traces. These are two counts in two windows and neither is a statement about a trend. The density of the threading traces among the integers, and the densities of the three classes within them, are not determined here.

11.4 Fields, and two distances

The quartic field at the golden stage. The eigenvalue of the minimal class W_0 of $G(5)$ is $\varepsilon = \varphi + \sqrt{\varphi}$, of degree 4 with palindromic minimal polynomial

$$x^4 - 2x^3 - 2x^2 - 2x + 1, \tag{11.7}$$

It is a Salem number: two of its conjugates lie exactly on the unit circle. The field generated by the endpoints of its axis is

$$L \;=\; \mathbb{Q}(\sqrt{\varphi}) \;=\; \mathbb{Q}[x]/(x^4 - x^2 - 1), \tag{11.8}$$

of degree 4 over $\mathbb{Q}$, containing $\mathbb{Q}(\sqrt{5})$, of discriminant -400, and with $r_1 = 2$, $r_2 = 1$ — not totally real. The mechanism is a single sign: under $\sigma : \varphi \mapsto (1-\sqrt{5})/2$ one has $\sigma(t^2 - 4) = 4\varphi' < 0$, so the second real place of $\mathbb{Q}(\sqrt{5})$ becomes complex in L. Theorem 2.10 closes the list of groups of this neighborhood whose *trace field* is quadratic; the fields attached to their classes are not all quadratic, and L is the first field met at a minimal class that is neither. The atlas of §2 indexes quadratic surds by their seats and has no sheet for occurrences of this kind, and no classification of them is attempted here.

The half-regulator. Proposition 5.13 gives

$$d\big(i,\ \text{the systolic axis of } G(5)\big) \;=\; \tfrac{1}{2}\ln\varphi \;=\; \tfrac{1}{2}\operatorname{Reg}\mathbb{Q}(\sqrt{5}) \tag{11.9}$$

exactly, and Remark 5.14 says what the value is and is not: a distance from a cone point to a geodesic, not a member of the length spectrum. By Corollary 5.9 the golden Hecke group does not carry the golden regulator, and Proposition 5.10 finds no rung $2k\ln\varphi$ of the golden ladder among its lengths, at the scope stated there. The regulator of $\mathbb{Q}(\sqrt{5})$ therefore appears here as a distance in a group that does not carry it as a length. Whether (11.9) has a mechanism, or is a coincidence of two closed forms, is not known.

The distance to the order-5 orbit. The corresponding closest approach of the same axis to the order-5 cone orbit is 1.15935676611550306 ..., with

$$\cosh d \;=\; \frac{1+\varphi}{2\sqrt{\varphi}\,\sin(\pi/5)}, \tag{11.10}$$

and unlike (11.9) it has not been put in closed form; the value appears to need an identity in $\mathbb{Q}(\zeta_{20})$, and none is offered here.

11.5 Two extensions catalogued and not carried out

Products of two parabolics. Proposition 10.1 indexes the parabolic classes of $X(1)$ by T^n with $n \neq 0$, and Corollary 10.3 leaves exactly two of them primitive. The minimal hyperbolic element is a product of one letter from each: writing $T' = ST^{-1}S^{-1}$, a parabolic fixing 0 and lying in the class of T^{-1} by Proposition 10.1,

$$T' = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \qquad T\,T' = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}, \qquad T\,T'^{-1} = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}, \tag{11.11}$$

the last two of trace 3 and 1. The two shortest nontrivial products of two parabolics fixing ∞ and 0 thus land on the two integer neighbours of the parabolic trace $t = 2$: the shortest closed geodesic above it, and an elliptic element of order 3 below it. Which hyperbolic classes of $X(1)$ are products of two parabolics fixing distinct cusps, and whether a word length in parabolic letters orders the trace values, is not investigated here; §10 does not depend on the answer.

Mutual exclusion outside the Hecke family. Theorem 9.5 forbids the combination "λ_q a unit and $G(q)$ arithmetic" under the hypothesis that $\mathbb{Q}(\lambda_q)$ is quadratic, and Theorem 2.10 makes that hypothesis hold at exactly three values of q; the proof is three steps and rests on no enumeration, so the three groups are the whole of the relevant population and not a sample of it. The same configuration can be posed for other triangle-group families with quadratic invariant trace field — the $(2, 3, q)$ groups are the natural next one — and there the question is whether the exclusion continues to hold. Such an instantiation would enlarge the population on which the phenomenon is observed rather than verify Theorem 9.5, and it is not carried out here.

Fifteen questions, then, at the boundary of a domain whose interior is settled. None of them is a gap in a result above: every statement of §§2–10 is made at the scope at which it holds, and the questions of this section begin where that scope ends.

Appendix A. Verification

Every statement of this paper that carries a range, a population or a count was checked by computer in exact arithmetic, and the checks are collected here rather than distributed through the text. The body sections state their bounds inline, as they must; this appendix states the discipline under which the bounds were produced, tabulates what was verified and how, and says what the verification does not establish. Nothing below is offered as evidence for a theorem whose proof is given in the body: where a result is proved, the computation is a control on the proof and is labelled as one.

A.1 The discipline

Exact arithmetic. All matrix, form, cycle, Pell and trace computations were carried out in exact integer arithmetic, or in exact arithmetic in the relevant quadratic ring — $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\varphi]$ were represented as pairs of integers with the defining relation applied at each product. Identities so computed have residual exactly zero. Floating-point arithmetic appears only in two roles, both declared at the point of use: reporting a decimal expansion, and bounding a geometric scan. Where a transcendental quantity enters an identity the residual is quoted with the identity; the largest residual quoted anywhere in this paper is 4.4×10^{-16}, at Propositions 6.1 and 7.9.

Repetition and byte comparison. Each computation was run twice and the two outputs compared byte for byte, on an unmodified instrument — a computation that has been altered between its two runs is not the same computation, and none was. The comparison was carried out and recorded for every computation reported in Tables A.1–A.10 **except** those marked (†). The (†) rows were computed in exact arithmetic, with their populations declared and their invariants checked and their residuals quoted, but the record of them does not state a re-run under byte comparison, and they are reported here as what they are. Sixteen rows carry the mark, together with one cross-check in Table A.6; they are exactly the entries drawn from the probes of $X(1)$ and $G(4)$, and they lie in Tables A.6, A.7, A.8 and A.10.

Checking against stated invariants. Every constructed object was checked against invariants fixed in advance of the construction rather than read off it. For the elements produced by Proposition 3.1 and used again at Theorem 3.6(b) and Theorem 8.1, the invariant triple is

$$\det N^2 = 1, \qquad \operatorname{tr} N^2 = t, \qquad N^2_{21} \equiv 0 \pmod{p}, \tag{A.1}$$

and every instance was checked against all three. The same practice governs the determinant and trace checks recorded throughout Tables A.2–A.10.

Controls before assertions. Each computation was first run as a control reproducing a result established independently of it, and only then over the range in which it was to be believed. Proposition 3.9's instrument was run at $t = 3$, where the trace equation has the unique solution $(Q, n) = (5, 1)$, before being run at $t = 6$; Proposition 7.6's was run over $3 \le t \le 40$, where the threading set was already known, before being run over $3 \le t \le 200$; Proposition 6.10's list of levels was checked against a list obtained by a direct scan. The controls are not a formality: more than one of them disagreed on its first run with the result it was reproducing, and in each such case the disagreement was traced to the instrument, or to the prior statement, before anything resting on it was recorded.

The narrow class numbers of Table 2.2 were recomputed by a routine sharing no code with the one that produced them, against a five-value control — h^+ at $D = 5, 8, 12, 13, 21$ — fixed before that routine existed.

Populations, and the two kinds of bound. Every quantified claim carries its population in the sentence that states it, and two structurally different kinds of population occur. A *ball* bounds word length in a fixed generating set: it is a sample of the group, and a statement made over it is silent about everything outside it. A *syllable-depth bound* in the free-product normal form (5.17) bounds the length of a cyclically reduced representative; since the trace is a conjugacy invariant, such a bound is an exhaustive statement about conjugacy classes up to that depth, and not a sample. The distinction is not cosmetic — it is why the census of Remark 5.7 is stated over 22,369,620 words and 182,474 distinct traces rather than over a ball radius, and why the earlier ball of 34,942 elements recorded in Corollary 5.6's verification is reported as an independent bounded check and not as the same statement.

Cross-checks. Where two computations of the same quantity were carried out independently of one another, both are reported and the fourth column of each table below names the second. One such agreement is a check on the enumerations themselves: the two independent enumerations of $G(4)$ used in §8 are a ball of 21,214 distinct elements at word length ≤ 14 and a ball of 6,268 at word length ≤ 12, and the level sizes of the first sum to exactly 6,268 through length 12.

Instrument tallies. Table A.1 records the declared check counts. Each count is the full arithmetic tally of the computation that produced the statements in the column beside it, not a count of checks bearing on any one of them; where a body section quotes a figure of the form "n checks, n pass, 0 fail" — at Propositions 2.8, 3.9 and 3.11 — that figure is the tally of the whole computation, and some of its checks bear on statements this paper does not make.

Table A.1. *The computations, by the statements they serve.*

computation	statements served	checks	passed	re-run and byte-compared
Hecke-group trace sets and systoles	§5	19	19	yes
the descent criterion and the floor-is-systole predicate	§§3, 6	13	13	yes
the bead ladder and the normalizer sweep	§§3, 6	11	11	yes
the two $\sqrt{2}$ seats as consecutive rungs	§2	10	10	yes
the threading classification	§7	8	8	yes
two computations outside this paper's domain	—	12	12	yes
the trace line of $X(1)$ and its class numbers	§2	32	32	yes
the trace equation and the level sweep	§3	10	10	yes

computation	statements served	checks	passed	re-run and byte-compared
the bead-scope bound	§6	5	5	yes
the realized trace set of $G(4)$, Type ↔ coset	§8	11	11	yes
two further computations outside this paper's domain	—	16	16	yes
the disc-5 and disc-8 comparison	§§2, 4, 5, 6, 7, 8	—	—	5 of 5 at close
the $G(5)$ probe, first channel	§§5, 7	—	—	13 of 13
the $G(5)$ probe, second channel	§5	—	—	7 of 7
the $X(1)$ and $G(4)$ probes	§§6, 7, 8, 10	—	—	(†)

The two groups of seven and of six computations totalling 73 and 74 checks respectively returned 73 and 74 passes and no failure. One byte comparison in the disc-5 and disc-8 group required a second attempt, the first having produced an empty output file; the re-run was byte-identical, and the incident is recorded here rather than absorbed. A further group of computations — those behind Propositions 2.1 and 2.2, Remark 2.3, Proposition 4.9, Theorem 9.1 and the primitive counts of §10 — records its checks individually rather than as a tally; those figures appear in the tables of §A.2 and are not repeated here.

A.2 The verified statements, by section

The *kind* column takes one of four values. **Exhaustive enumeration** means the stated population was traversed completely; whether that population is the whole object or a bounded box is stated in the population column and taken up again in §A.3. **Constructed witness** means an object with the asserted properties was exhibited and checked against its invariants. **Identity check** means an equality was evaluated on both sides. **Independent re-derivation** means the quantity was obtained a second time by a route sharing no step with the first.

Table A.2. *§2 — the atlas and the organizing theorem.*

statement	population or range	kind	cross-check
Proposition 2.1, both halves of (2.7)	every order $2 \le m \le 12$ — 11 orders; 50 decimal places, tolerance 10^{-40}	identity check	the full product (2.6) and the closed form (2.7) evaluated separately, 11 of 11 each
Remark 2.3, the coincidence (2.11)	4 checks	identity check	the uniqueness is proved outright; the checks are a control on it, not its evidence

statement	population or range	kind	cross-check
Table 2.2, the narrow class numbers	all ten hyperbolic rows, $0 \le t \le 12$; 32 checks	independent re-derivation	recomputed from reduced-form cycles by a routine sharing no code with the one that produced the table, against a five-value control fixed before that routine existed
Proposition 2.8, the two rungs (2.16)–(2.17)	10 checks	identity check	—
Theorem 2.10, the degree (2.19)	every q with $3 \le q \le 200$, exhaustive	exhaustive enumeration	returned $\{4, 5, 6\}$ and nothing else, 0 failures

Table A.3. *§3 — the squaring correspondence and the level law.*

statement	population or range	kind	cross-check
Proposition 3.1, the construction (3.5)–(3.6)	every integer trace $3 \le t \le 80$ — 78 traces	constructed witness	each instance checked against (A.1); 0 failures
Theorem 3.3, the identities (3.11)–(3.14)	all 28 fundamental discriminants $5 \le D \le 204$ with $N(\varepsilon_D) = -1$, exhaustive; the units found by a search bounded at $v \le 3 \cdot 10^5$	exhaustive enumeration	the population and eleven of its rows (u, v, t, p, n), including $D = 73$ with $(u, v) = (2136, 250)$, reproduced by two later computations written independently of the first
Theorem 3.6(a), necessity	$3 \le t \le 30$, complete in (b, c) over all divisor pairs for $\lvert a\rvert \le 30$; the descent set saturated, $\lvert a\rvert \le 10$ and $\lvert a\rvert \le 30$ returning identical counts	exhaustive enumeration	no descending class violates $n \mid$ content and no class with $n \mid$ content fails to descend
Theorem 3.6(b), sufficiency	$3 \le t \le 200$: 4,237 classes examined by exhaustive reduced-form cycles, 2,464 of them with $n \mid$ content, a descent constructed for every one	constructed witness	each N^2 checked against (A.1); the symbolic identities of (a) re-verified on every admissible (a, b, c, d, p) in $[-3, 3]^4 \times \{2, 3, 5, 6, 7, 10\}$; 0 failures

statement	population or range	kind	cross-check
Proposition 3.9, the trace equation and its witnesses	$Q \le 500$, $n \le 59$ for (3.37); levels $N \le 60$, $Q \parallel N$, matrix entries of absolute value at most 12 for the group search — exhaustive over that box; 10 checks	exhaustive enumeration	the instrument shares no code with Theorem 3.3's, and was run first at $t = 3$, where it reproduced the unique solution $(5, 1)$ and its witness (3.40)
Proposition 3.11, one geodesic at two levels	11 checks	identity check	—
Proposition 3.12, the normalizer sweep	levels $N \le 60$, exhaustive; a brute-force control at the twelve levels 4, 8, 9, 12, 16, 18, 25, 27, 36, 48, 49, 50 over $\lvert a\rvert, \lvert b\rvert, \lvert d\rvert \le 6$	exhaustive enumeration	the control sits on top of the conjugation derivation of (ii), not in place of it
Proposition 3.13, the degree criterion	every integer trace $3 \le t \le 200$	exhaustive enumeration	0 mismatches

Table A.4. *§4 — the single-inequality dichotomy.*

statement	population or range	kind	cross-check
Proposition 4.9, $\lambda_q^2 \in \mathbb{Q} \iff q \in \{3, 4, 6\}$	every q with $3 \le q \le 12$	independent re-derivation	the list was re-instantiated rather than only cited, and matches the published classification

Table A.5. *§5 — systoles of the floors.*

statement	population or range	kind	cross-check
§5.1, the coset grading (5.4)	$G(4)$: all cyclically reduced words of syllable length ≤ 8, exhaustive, 335 distinct traces. $G(6)$: syllable length ≤ 7, exhaustive, 2,125 distinct traces	exhaustive enumeration	—

statement	population or range	kind	cross-check
§5.2, the least integer trace of $\Gamma_0(2)$ above 2	brute force over $\lvert a\rvert, \lvert d\rvert \leq 40$, returning $\{4, 6, 8\}$ in $(2, 8]$	exhaustive enumeration	an exact trace-set computation on the floor independently returns $4, 6, 8, 10$ for $3 \leq t < 12$ and 4 as the least; neither uses the criterion of Lemma 5.2(a)
Theorem 5.5, the minimal-trace ladder	all 4^k cyclically reduced words at syllable count $k = 1, \ldots, 11$ — 5,592,404 words	exhaustive enumeration	a second, independently written computation reproduced the minimizing element (5.24) with its length, eigenvalue, endpoint field and cone count; it does not address minimality, and no part of Theorem 5.5 rests on it
Corollary 5.6, $\sqrt{5}$ and 3 unrealized	11,661 distinct realized trace values, syllable count $k \leq 9$	exhaustive enumeration	an earlier and independent ball of 34,942 elements at syllable length ≤ 14, 408 distinct traces, returned no element of trace $\sqrt{5}$, and recorded that this was not a proof
Remark 5.7 and Proposition 5.10, the integer traces $\{0, \pm 2\}$	all cyclically reduced words of syllable length ≤ 12 — exhaustive over conjugacy classes, 22,369,620 words, 182,474 distinct traces	exhaustive enumeration	—
Proposition 5.12, the cone points miss the axis	every elliptic fixed point in a ball at $q = 5$ — 595 points	exhaustive enumeration	the statement for the two home cone points is exact; only the orbit statement is bounded by the ball
Proposition 5.13, the closest approach	a geometric scan of one period at 40,000 sample points	independent re-derivation	matches the exact value (5.42) to residual 4.70×10^{-10}; the two computations share no step

Table A.6. *§6 — the doubling law and the bead ladder.*

statement	population or range	kind	cross-check
Proposition 6.1, $2 \operatorname{arccosh} 3 = 4 \ln(1+\sqrt{2})$ (†)	—	identity check	residual 4.4×10^{-16}
Proposition 6.1, length against regulator (†)	the four traces $t = 3, 4, 5, 6$	independent re-derivation	computed on the trace line rather than on the floor: $\ell = 4 \cdot (\text{regulator})$ at $t = 3, 6$ and $2 \cdot (\text{regulator})$ at $t = 4, 5$
Proposition 6.2, $\mathcal{O}_{32}$ has no unit of norm -1 (†)	exhaustive over f; the Pell sweep (6.11) over $y \leq 10^5$, no solution	exhaustive enumeration	the powers $(1,1), (3,2), (7,5), (17,12)$ of ε_8 in exact integer arithmetic; the impossibility itself is proved at (6.12)
Remark 6.3, the $t = 3$ companion (†)	—	identity check	—
Proposition 6.7, the pivots	the printed pivot lists	independent re-derivation	computed from the other side by a route sharing no step (†) — the on-axis condition $(a-c)^2 - 2c^2 = -1$ with a sweep over $c \leq 10^5$ returns the same three coefficient pairs $(1,1), (7,5), (41,29)$ in the same order
Theorem 6.8, the ladder	$\lvert a \rvert \leq 4000$, exhaustive via the Pell reduction: 36 axis involutions at $p = 5$ with 35 distinct coordinates covering 35 consecutive k, $-16 \leq k \leq 18$; 21 at $p = 2$ with 21 distinct coordinates covering 21 consecutive k, $-10 \leq k \leq 10$	exhaustive enumeration	agrees on the overlap with the earlier and independent enumeration of Observation 6.4, run from a different parametrisation with a search box of ± 60, which returned 19 and 11 consecutive k
Proposition 6.10, the floor-is-systole criterion	the 28-member population: 4 true positives, 0 false positives, 0 false negatives, 24 true negatives, the positives being $D \in \{5, 8, 13, 53\}$	exhaustive enumeration	a second control, free to disagree with the first, ran the criterion over every squarefree p with $2 \leq p \leq 20{,}000$ and returned exactly the 25 levels (6.26), reproducing a list obtained independently by a direct level scan

Table A.7. *§7 — reciprocity and threading.*

statement	population or range	kind	cross-check
Theorem 7.1, the identities (7.2)–(7.3) (†)	306 sampled matrices of $\mathrm{SL}(2,\mathbb{Z})$ — 30 symmetric, 276 not	identity check	0 failures
Proposition 7.2, two pivots per period (†)	every reciprocal class of trace $t \leq 40$	constructed witness	over the same range the number of representations of $t^2 - 4$ in the form (7.13) is in every case exactly twice the number of reciprocal classes at that trace
Theorem 7.4, the threading criterion (†)	every integer trace $3 \leq t \leq 40$, exhaustive in (s, b)	exhaustive enumeration	the criterion was obtained independently by a second derivation reaching (7.12) from (7.3) and (7.5) and no other input
Theorem 7.5, the norm-(-1) family threads	$u = 1, \ldots, 12$	constructed witness	the predicted set $\{3, 6, 11, 18, 27, 38\}$ is contained in the threading set $\{3, 6, 7, 11, 15, 18, 27, 34, 38, 39\}$ computed independently over $3 \leq t \leq 40$
§7.2, the threading classes at $t = 7$ and $t = 34$	—	independent re-derivation	identified at word level as $(LR)^2$ and $(L^2R^2)^2$, by a route independent of the level-selection identity (3.11)
Proposition 7.6, the classification of threading traces	$3 \leq t \leq 200$, exhaustive	exhaustive enumeration	the instrument was first run as a control over $3 \leq t \leq 40$, where it reproduced the independently computed threading set exactly
Proposition 7.9(i)–(iii), the $G(4)$ instance (†)	—	identity check	exact in $\mathbb{Z}[\sqrt{2}]$; the length (7.25) matches $2 \operatorname{arccosh} \sqrt{2}$ to residual 4.4×10^{-16}
Proposition 7.9(iv), the axis pattern (7.26) (†)	the ball of word length ≤ 12 — 6,268 distinct elements, yielding 3,135 order-4 and 4,061 order-2 orbit points; the 18 minimal-trace elements of word length ≤ 8	exhaustive enumeration	conjugators to A or A^{-1} exhibited for all 18; the pattern held to six decimals throughout

statement	population or range	kind	cross-check
Proposition 7.10, the $G(5)$ census	the 32 classes of syllable length $m \leq 3$; threading tested at 40 digits with threshold 10^{-20}	exhaustive enumeration	reciprocity is decided exactly from the word and carries no bound; only the threading column's negative half is bounded by the ball
Proposition 7.12(i), the non-reciprocal witness	a ball of 152,914 elements — all reduced words of syllable length ≤ 15	exhaustive enumeration	0 inverting conjugators found
Theorem 7.13, the archimedean implication (7.29)	five classes registered in advance, and 3,360 cyclic classes of syllable length $n \leq 7$ — 3,346 hyperbolic, 42 reciprocal	exhaustive enumeration	all 42 satisfy (7.29), 0 violations
Remark 7.14, the interface identity (7.31)	the five registered classes	identity check	—

Table A.8. *§8 — the realization laws and the trace gaps.*

statement	population or range	kind	cross-check
Theorem 8.1, the hyperbolic realization	every integer trace $3 \leq t \leq 80$ — 78 traces (the computation of Table A.3's first row)	constructed witness	checked against (A.1); 0 failures
Theorem 8.1, the class census at $t = 3, \dots, 6$ (†)	$t = 3, \dots, 6$	independent re-derivation	obtained by a word census independent of the form-cycle route that produced Table 2.2
Theorem 8.2, the closed form (8.10) (†)	all 21,214 distinct elements of word length ≤ 14, zero exceptions	exhaustive enumeration	the containment half separately derived on a second enumeration of 6,268 distinct elements at word length ≤ 12, which printed the realized ladder and found no odd integer and nothing outside $2\mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$; the two balls agree exactly in size through length 12
Proposition 8.3, Type $\leftrightarrow$ coset (8.16)–(8.17)	symbolic in (a, b, c, d) and carrying no bound; controlled on the witness families (8.13)–(8.14) over $k, m \in [-4, 8]$	identity check	a bounded control on an unbounded derivation: every image of the right shape, determinant 1, trace preserved

statement	population or range	kind	cross-check
Corollary 8.5(iii), the two intervals (†)	the word-length-$\leq$ 14 ball	exhaustive enumeration	count 0
Proposition 8.8(a), the seam law (8.24) (†)	$\delta = 10^{-1}, 10^{-2}, 10^{-4}, 10^{-6}$	identity check	at $\delta = 10^{-2}$ the surd is 0.199749844 against $2\sqrt{\delta} = 0.200000000$
Proposition 8.8(b), the enumerated clause (†)	the five probed values 0.5, 1.5, $\sqrt{2}$, $\sqrt{3}$, 1.99	exhaustive enumeration	the support statement (8.25) itself follows from (8.6) and carries no bound; only the enumerated clause is bounded

Table A.9. *§9 — the value identity and the mutual-exclusion theorem.*

statement	population or range	kind	cross-check
Theorem 9.1, the identity (9.5)	13 independent closed-form checks	independent re-derivation	a fundamental-unit routine called at discriminant 5 returns $(u, v, \mathrm{den}, N) = (1, 1, 2, -1)$ — that is, $\frac{1}{2}(1+\sqrt{5})$ of norm -1 — by a route that does not assume the identity; two comparisons in the original tally compared a value with itself and are excluded from the count of 13
Proposition 9.7, the unit rule (9.18)	$q = 3, \dots, 30$, exhaustive, 0 failures	exhaustive enumeration	the consequence (9.19) checked separately; over the same range the even units are $q = 12, 20, 24, 28, 30$, which is what falsifies the parity form of the rule
(9.20), the quadratic trace fields	$q = 3, \dots, 200$	exhaustive enumeration	0 failures; the same computation as Table A.2's last row

Table A.10. *§10 — the parabolic trace and the primitivity filter.*

statement	population or range	kind	cross-check
Proposition 10.1, the conjugation formula (10.1) (†)	all 116 determinant-one integer matrices with entries of absolute value at most 3, and $n \in \{1, 2, -1, 5\}$	identity check	—
Proposition 10.1, no cross-conjugacy (†)	all $g \in \mathrm{SL}(2, \mathbb{Z})$ with entries of absolute value at most 8, and $n, m \in [-3, 3] \setminus \{0\}$	exhaustive enumeration	returned exactly the diagonal pairs (n, n) with the $+$ sign and no cross pair; the search bounds the conjugators only, and Step 2 of the proof closes the statement for all of them
Theorem 10.5 and the table beneath it, the primitive counts	$t = 3, \dots, 12$	independent re-derivation	computed twice by structurally unrelated routines — the power-trace recursion (10.5) and an independent enumeration of reduced-form cycles — agreeing at every trace

A.3 What the verification does not establish

A byte comparison establishes reproducibility, not correctness. The two are independent, and the separation was met in this work: a point-to-geodesic distance formula that is wrong away from the centre line reproduced itself exactly under re-running, and was caught not by the comparison but by a control whose answer was known in advance — a distance that had to come out 0 and came out 0.4812. Every claim of this appendix about repetition should be read in that light: it certifies that a computation is what it says it is, and nothing about whether the thing it computes is the thing that was wanted. The invariant checks of (A.1) and the controls of §A.1 are what bear on correctness, and they bear on it only to the extent that the invariants and the controls were fixed independently.

A bounded enumeration bounds its box. Every population in Tables A.2–A.10 is a recall bound except where it is stated to be exhaustive over the whole object, and a recall bound is silent outside itself: the enumeration is complete inside the box and says nothing beyond it. Three consequences are worth naming. A ball is additionally a *sample* of the group rather than a statement about its conjugacy classes, so the ball-based rows of Tables A.5–A.8 are weaker in kind than the syllable-depth rows beside them. A statement proved in the body and controlled by a computation here does not inherit the computation's bound — Theorems 3.6, 5.5, 6.8, 8.2, 8.10 and 8.12 are proved for all of their ranges, and the corresponding table rows are controls. And a statement that rests only on a bounded enumeration is reported as such at the point where it is made: Proposition 5.10's even-k non-realizations at $k \geq 4$, Proposition 7.10's negative column, and Corollary 8.7's exclusion of mixed traces at stages other than $G(4)$ are the three places in this paper where that applies.

Enlarging a box would not settle the open case. The census of Remark 5.7 — that the realized integer traces of $G(5)$ are exactly $\{0, \pm 2\}$ over all cyclically reduced words of syllable length ≤ 12

— is not a statement that a larger box would convert into a theorem: the congruence conditions the enumeration itself suggests do not separate, since at syllable length ≤ 8 the four primes inert in $\mathbb{Q}(\sqrt{5})$ jointly leave 32 of 87,380 classes surviving rather than none, and the condition at 2 admits 68,758 of the 182,471 distinct non-integer trace values into the same stratum as the integer-trace ones. What would settle it is an exact statement about the φ-coefficient of the trace as a function of the word, which §11.1 names as an edge of this paper's domain, in the paragraph following Proposition 11.1.

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